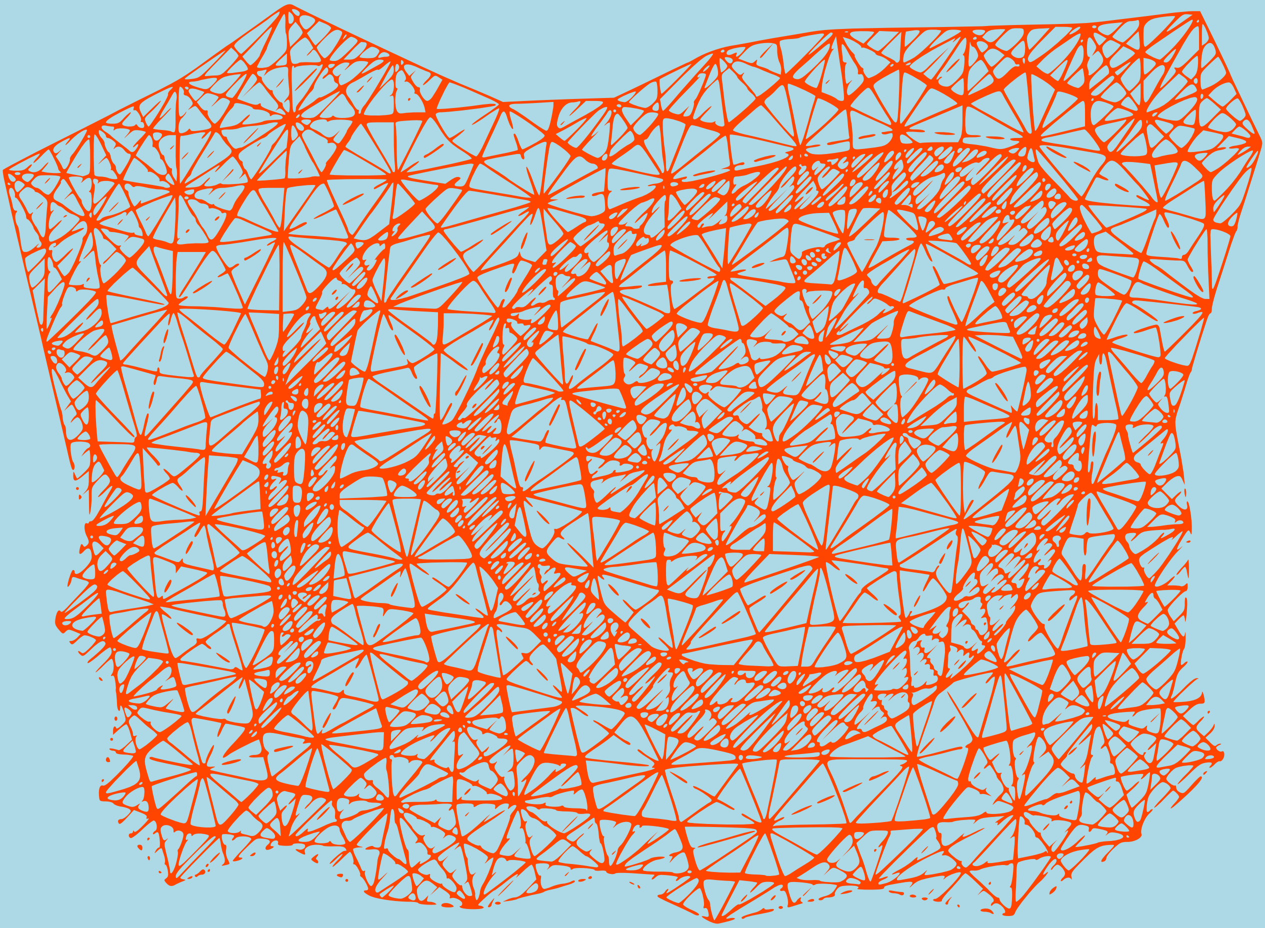
P.S. Aleksandrov

# COMBINATORIAL TOPOLOGY

## Volume 3



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*VOLUME 3*

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# COMBINATORIAL TOPOLOGY

## VOLUME 3

*BY*

P. S. ALEKSANDROV

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# CONTENTS

## PART FOUR

## HOMOLOGICAL MANIFOLDS. THE DUALITY THEOREMS. COHOMOLOGY GROUPS OF COMPACTA

*Chapter*

XIII. Homological Manifolds ($h$-Manifolds)

1. Definition and Simplest Properties .......... 4
    1.1. Definition of $h$-manifolds .......... 4
    1.2. Elementary properties of $h$-manifolds .......... 4
    1.3. The case $n \leq 3$ .......... 6
    1.4. Barycentric stars and $h$-manifolds .......... 8
2. The Barycentric Complex of a Combinatorial $h$-Manifold .......... 9
    2.1. Notation; preliminary basic facts .......... 9
    2.2. The complex $\mathfrak{K}^*$ .......... 10
3. The Intersection Number, the Isomorphism $D^q$ and the Poincaré Duality .......... 11
    3.1. The intersection number $(t_i^p \times *t_i^q)$ .......... 11
    3.2. The intersection number and the incidence numbers .......... 14
    3.3. The isomorphism $D^q$ and the Poincaré duality .......... 17
    3.4. The intersection number $(x^p \times *x^q)$ .......... 21
    3.5. Veblen's theorem .......... 23
4. The Combinatorial Form of the Alexander Duality .......... 24
    4.1. Formulation of the theorem .......... 24
    4.2. Generalization of Theorem 4.10 .......... 26
    4.3. The cases $p = 0$ and $p = n - 1$; prefatory remarks to the proof of Theorem 4.2 .......... 29
    4.4. The isomorphism $\Delta D^q$ .......... 30
    4.5. Definition and simplest properties of the linking number $\mathfrak{v}(u^p, u^{q-1})$ .......... 34
    4.6. Theorem on linked systems of cycles .......... 37

XIV. Cohomology Groups of Compacta and the Alexander-Pontryagin Duality

1. Cohomology Groups of Bicompacta .......... 41
    1.1 Spectra and their cohomology groups .......... 41
    1.2. The homomorphism $\tilde{S}^\alpha$ .......... 44
    1.3. The cohomology groups of bicompacta .......... 44
    1.4. The group $\nabla^{00}$ .......... 45
2. Cofinal Suborders of Spectra. The Case of Compacta .......... 46
    2.1. Cofinal suborders of spectra .......... 46
    2.2. The case of $\Phi$ a compactum .......... 47
    2.3. The case of $\Phi$ a polyhedron .......... 48

*Chapter*

3. The Homology Groups of Open Subsets of $S^n$. Formulation of the Alexander-Pontryagin Duality and Its Fundamental Consequences. Finite Compacta . . . . . . . . . . . . . . . . . . . 50
3.1. The group $\Delta^r(\Gamma)$, where $\Gamma$ is an open subset of $S^n$ . . . . 50
3.2. The barycentric subdivision of a chain of $\Gamma$ . . . . . . . . . . 50
3.3. The topological invariance of the groups $\Delta^r(\Gamma)$ . . . . . . 51
3.4. The Alexander-Pontryagin duality and its fundamental consequences . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 52
3.5. Special case: $\Phi$ is finite . . . . . . . . . . . . . . . . . . . . . . . . . . 54
4. The Complexes $K_\alpha$, $K_{0\alpha}$, $K_\alpha^*$, $K_{0\alpha}^*$, $Q_{\alpha 1}$; the Sets $G_\alpha$ and $\Gamma_\alpha$ . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 56
4.1. Definitions and notation . . . . . . . . . . . . . . . . . . . . . . . . . . 56
4.2. The continuous mapping $C$ of $G_\alpha$ onto $\| Q_{\alpha 1} \|$ . . . . . . . . 59
4.3. Some properties of the group $Z^r(\Gamma)$ . . . . . . . . . . . . . . . . 59
5. A Combinatorial Lemma . . . . . . . . . . . . . . . . . . . . . . . . . . . 60
5.1. Introductory remarks and formulation of the lemma . . 60
5.2. The fundamental identity . . . . . . . . . . . . . . . . . . . . . . . . 61
5.3. The completion of the proof of Lemma 5.1 . . . . . . . . . . . 64
6. The Spectrum $\mathcal{K}_0$ of the Set $\Phi$ . . . . . . . . . . . . . . . . . . . . . . . 66
6.1. The coverings $\Omega_\alpha$ and $\omega_\alpha$ . . . . . . . . . . . . . . . . . . . . . . . . 66
6.2. The spectrum $\mathcal{K}_0$ of the compactum $\Phi$ . . . . . . . . . . . . . . 67
6.3. The projections $S_\alpha^\beta$ . . . . . . . . . . . . . . . . . . . . . . . . . . . . 68
7. Further Auxiliary Propositions . . . . . . . . . . . . . . . . . . . . . . . 68
7.1. Commutativity of the operators $\bar{S}_\beta^\alpha$ and $\nabla E$ . . . . . . . . . 68
7.2. Corollary to (5.1) and (7.1) . . . . . . . . . . . . . . . . . . . . . . . 69
8. Proof of the Alexander-Pontryagin Duality . . . . . . . . . . . . . . 70
8.1. The homomorphism $\Delta D^q$ of $\nabla^p(\Phi)$ into $\Delta^{q-1}(\Gamma)$ induced by the homomorphisms $\Delta D_\alpha^q$ . . . . . . . . . . . . . . . . . 70
8.2. The mapping $\Delta D^q$ is an isomorphism . . . . . . . . . . . . . . . . 71
8.3. The isomorphism $\Delta D^q$ is onto . . . . . . . . . . . . . . . . . . . . 71

XV. LINKING. THE LITTLE ALEXANDER DUALITY

0. Introductory Remarks . . . . . . . . . . . . . . . . . . . . . . . . . . . . 73
1. The Intersection Number and the Linking Number . . . . . . . 73
1.1. The intersection number of two chains $x^p$ and $y^q$ of $R^n$ . 73
1.2. Computation formulas for intersection numbers . . . . . . 74
1.3. Corollaries of Theorem 1.2. Generalization of the definition of intersection number . . . . . . . . . . . . . . . . . . . . . . . 77
1.4. The linking number . . . . . . . . . . . . . . . . . . . . . . . . . . . . 80
1.5. Computation formulas for the linking numbers . . . . . . . 82
1.6. The case $z^p \in Z_\Delta^p(K_0)$, $*z^{q-1} \in Z_\Delta^{q-1}(K^*)$ . . . . . . . . . . . 83
2. Linking of Proper Cycles . . . . . . . . . . . . . . . . . . . . . . . . . . 85

*Chapter*

2.1. Definition of linking numbers for proper cycles.. 85
2.2. Deformation theorem for singular cycles.............. 87
3. The Little Alexander Duality.......................... 88
3.1. Introductory remarks and formulation of the theorem. 88
3.2. Proof of Theorem (3.1)........................... 89
3.3. The little Pontryagin duality...................... 91
3.4. Closed $(n-1)$-dimensional pseudomanifolds in $S^n$ $(n \geq 2)$. The Jordan-Brouwer theorem ............. 93
3.5. Remarks on linking of pseudomanifolds.. 95

## PART FIVE

## INTRODUCTION TO THE THEORY OF CONTINUOUS MAPPINGS OF POLYHEDRA

XVI. THE BROUWER THEORY OF CONTINUOUS MAPPINGS IN $R^n$ AND $S^n$
1. Index of a Point Relative to an $(n-1)$-Cycle in $R^n$..... 100
1.1. The combinatorial case........................... 100
1.2. The index of a point $o \in R^n$ relative to a proper cycle $\mathfrak{z}^{n-1}$ ...................................... 101
1.3. The Poincaré-Bohl theorem and Rouché's theorem... 103
1.4. Homologies in $R^n \setminus o$......................... 103
1.5. The interpretation of the index of a point relative to a cycle as the degree of the central projection of the cycle onto a sphere.......................... 104
2. The Existence of Roots............................. 105
2.1. Existence theorem for roots........................ 105
3. The Local Degree of a Mapping of an $n$-Chain Into $R^n$.. 106
3.1. Definition and fundamental properties of the local degree.................................... 107
3.2. Simplicial approximations........................ 107
3.3. Mappings into $S^n$.............................. 109
4. Topological Mappings............................. 109
4.1. Theorem 4.1................................... 109
4.2. Invariance theorems............................. 110
5. Vector Fields and Continuous Mappings................ 111
5.1. The connection between vector fields and mappings... 111
5.2. The index of an isolated zero of a vector field........ 113
5.3. Vector fields on a solid sphere..................... 114
5.4. Brouwer's fixed point theorem for an $n$-cell.......... 115
5.5. Vector fields on spheres of even dimension........... 116

*Chapter*

5.6. Reflections of spheres; another fixed point theorem . . . 117
5.7. Exercises . . . 118
6. Classification of Continuous Mappings of an $n$-Sphere Into an $n$-Sphere . . . 119
6.1. Hopf's theorem; introductory remarks . . . 119
6.2. Mappings of degree zero . . . 122
6.3. Reduction of Hopf's theorem to an extension theorem. 124
6.4. Proof of the extension theorem . . . 125
XVII. FIXED POINTS OF CONTINUOUS MAPPINGS OF POLYHEDRA
1. The Fixed Point Theorem . . . 128
1.1. Fixed simplexes . . . 128
1.2. Hopf's formula . . . 129
1.3. The Lefschetz number of a continuous mapping of a polyhedron into itself . . . 131
1.4. Examples . . . 132
1.5. Some remarks on the Lefschetz number . . . 133
2. The Index of a Fixed Point . . . 134
2.1. Definition of the index . . . 134
2.2. Some properties of the index . . . 135
2.3. Normal fixed points. Topological invariance of the index of a fixed point . . . 136
2.4. Fixed points of affine mappings . . . 138
3. The Algebraic Number of Fixed Points of a Continuous Mapping of a Polyhedron Into Itself . . . 138
3.1. Definition of regular fixed points. Formulation of the fundamental theorem . . . 138
3.2. Generalization of Theorem 3.1 . . . 139
3.3. Regular fixed points of simplicial mappings . . . 139
3.4. Reduction of Theorem 3.1 to the approximation theorem . . . 140
3.5. Proof of the approximation theorem . . . 141
3.6. Remarks on the algebraic number of fixed points. . . . 144
REFERENCES . . . 146
INDEX . . . 147

*Part Four*

# HOMOLOGICAL MANIFOLDS. THE DUALITY THEOREMS. THE $\nabla$-GROUPS (COHOMOLOGY GROUPS) OF COMPACTA

Polyhedra occupy an intermediate position in topology. For some purposes, the class of figures encompassed by polyhedra is too wide; for others, too narrow. It is too wide from the point of view of the wealth of concrete geometric facts and problems which are revealed when one passes from general polyhedra to the special case of *manifolds*, which have proved to be one of the fundamental and central objects of topological research. On the other hand, the class of polyhedra and their topological images proves to be too narrow and adventitious if one is interested in the more general concepts of modern topology. One then seeks a natural class of figures broad enough to allow these concepts, *without ceasing to be geometrical* (that is, without losing their original connection with our spatial intuition), to reveal the mathematical essence, in all its fullness and diversity, of this set of geometric objects. In short, more general concepts require a wider and more abstract field of application, and it is then necessary to include, at any rate, all the compacta and, apparently, even all the bicompacta (bicompact Hausdorff spaces). We wish to emphasize once more, however, that our object is not to generalize for the sake of generalization, but to understand the deeper laws governing a broad, difficult and diverse field of mathematical phenomena.

It is therefore not surprising that we discern two clearly distinguished directions in topology: the topology of manifolds and the topology of bicompact and locally bicompact Hausdorff spaces.

Each of these main fields has many branches; in this book we can consider only the simplest and most fundamental results of a few of these subfields. We discuss those parts of the theory of manifolds and bicompact spaces which can be treated by *homology* methods. The rigorous treatment of homology theory was begun in Chapter VII, and the methods introduced there and in subsequent chapters are used throughout the rest of the book. In this part of the book we shall see that both main branches of topology meet at one of the highest points common to both: in the Alexander-Pontryagin duality, which is the culmination of Part Four (see Chapter XIV).

The common parts of both branches (manifolds and bicompact spaces) may also be investigated further by means of the homology rings of locally bicompact spaces. Unfortunately, this subject is outside the scope of the book.

The homology theory of compacta, one of whose main adornments is the Alexander-Pontryagin duality, can be developed in two ways. One possibility is to transfer to compacta the fundamental notions of $\Delta$-cycle and $\Delta$-homology and to use these to define the $\Delta$-groups of compacta. We used this method in Chapters XI and XII; the full development of this method leads unavoidably to topological homology groups and to the use of bicompact topological groups as coefficient domains (see Pontryagin [a]* and Steenrod [a]).

The second method is to generalize the $\nabla$-theory to compacta and bicompacta. This method was first proposed by A. N. Kolmogorov. The variant of this theory which is developed in the book is closely related to the $\nabla$-theory of ordinary complexes.

In the case of topological spaces the decisive advantage of the $\nabla$-theory over the $\Delta$-theory is that the former does not require the use of topological groups. Although it appears that the $\nabla$-theory is more suitable to the investigation of figures more general than polyhedra, both theories are essentially equivalent, since they are dual to each other in the sense of the Pontryagin theory of characters.

Chapter XIV is devoted to the introduction of the fundamental concepts of the $\nabla$-theory and to the proof of the Alexander-Pontryagin duality. This proof depends on the combinatorial form of the duality (which is a theorem on subcomplexes imbedded in a combinatorial $h$-manifold) treated in Chapter XIII. The elements of the theory of $h$-manifolds is given in Chapter XIII. The first definition of $h$-manifolds given in that chapter is an invariant definition based on the concepts of Chapter XII. But this dependence is only apparent, because only a few lines later we introduce the concept of a *combinatorial h-manifold* (a notion independent of Chapter XII) and it is this concept alone which is used in the remainder of Chapter XIII. For this reason, Chapters XIII and XIV can be read without first reading Chapters XI and XII. In that case, however, XIV, 3.3 should be omitted.

The concept of $h$-manifold (introduced in 1926–1927 at approximately the same time by Alexander, Pontryagin and van Kampen) is the most natural generalization of manifolds for the purposes of homology theory: the definition singles out the local homology properties of ordinary manifolds—the homological homogeneity and simplicity of $h$-manifolds is more accessible to investigation than topological homogeneity in the large. In addition to the elementary theory of $h$-manifolds and the combinatorial form of the Alexander duality, Chapter XIII treats the Poincaré duality and the stronger form given it by Veblen.

In Chapter XV we deal with the more elementary form of the Alexander-

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* See the references listed at the end of the book.

Pontryagin duality, the duality as applied to curved polyhedra (in particular, the Jordan-Brouwer theorem). The exposition here depends only on Chapter XIII and on the simplest notions of Chapter XI; it is completely independent of Chapter XIV. The simpler form of the duality, which we call the "little" duality, is preceded by the theory of linking in Euclidean $n$-space. This theory belongs to the classical elementary parts of topology and should have its own interest for the reader.

As references for the questions discussed in Part Four we may cite, in addition to the paper of Pontryagin alluded to above: Aleksandrov [f (see vol. 1, Bibliography); k; l], Kolmogorov [a; b; c; d], Alexander [b], Steenrod [a]. These papers contain references to the earlier literature.

*Chapter XIII*

# HOMOLOGICAL MANIFOLDS ($h$-MANIFOLDS)

## §1. Definition and simplest properties

### §1.1 Definition of $h$-manifolds.

DEFINITION 1.1. *An $n$-dimensional homological manifold, or simply an $n$-dimensional $h$-manifold, is a connected $n$-dimensional polyhedron whose local integral Betti groups in each point are isomorphic to the local Betti groups of Euclidean $n$-space.*

In other words, an $n$-dimensional $h$-manifold is a connected $n$-dimensional polyhedron $\Phi$ with the property that for every $p \in \Phi$ the group $\Delta_p^n(\Phi, J)$ is infinite cyclic, while $\Delta_p^r(\Phi, J) = 0$ (the null group) for every $r < n$.

REMARK 1. We shall prove in Remark 3 that if $\Phi$ is an $n$-dimensional $h$-manifold, then $\Delta_p^r(\Phi, \mathfrak{A})$, $r < n$, is the null group and $\Delta_p^n(\Phi, \mathfrak{A})$ is isomorphic to $\mathfrak{A}$ for an arbitrary coefficient domain $\mathfrak{A}$.

Theorem 7.2 of XII implies

1.11. If $\Phi$ is an $n$-dimensional $h$-manifold, then every triangulation $K_\alpha$ of $\Phi$ has the following property:

1.111. *If $O_\alpha = O_{K_\alpha}T_\alpha$, $T_\alpha \in K_\alpha$, is an arbitrary star, the group $\Delta_0^n(O_\alpha)$ is infinite cyclic and $\Delta_0^r(O_\alpha) = 0$, $r < n$.*

REMARK 2. Hence it follows from XII, Theorem 7.2, Corollary 5 that every $(n - 1)$-simplex of $K_\alpha$ is incident with precisely two $n$-simplexes. Moreover, $K_\alpha$ is clearly a pure complex (VI, Def. 5.24). If a triangulation of a polyhedron $\Phi$ has property 1.111, so does every triangulation of $\Phi$; hence $\Phi$ is an $n$-dimensional $h$-manifold. Thus, the $n$-dimensional simplicial complexes satisfying 1.111, and only such complexes, are isomorphic to triangulations of $h$-manifolds; these complexes are called *combinatorial $h$-manifolds.*

In this chapter we shall have to do only with combinatorial $h$-manifolds and, therefore, the discussion will be essentially independent of XII. The invariance of the $h$-manifolds, however, follows from the results of XII.

REMARK 3. A consequence of IX, Theorem 4.1 is the following proposition: If $\Delta_0^r(O_\alpha) = 0$, $r < n$, and $\Delta_0^n(O_\alpha)$ is infinite cyclic, then $\Delta^r(O_\alpha, \mathfrak{A}) = 0$ for $r < n$ and $\Delta^n(O_\alpha, \mathfrak{A})$ is isomorphic to $\mathfrak{A}$. This implies the assertion in Remark 1.

REMARK 4. An immediate consequence of Def. 1.1 and XII, Def. 6.2 is that every $n$-manifold (see I, 5.3) is an $h$-manifold.

### §1.2. Elementary properties of $h$-manifolds.

1.21. *If $K$ is an $n$-dimensional combinatorial $h$-manifold and $e$ is a vertex* *of $K$, the star $O_Ke$ is an $n$-dimensional orientable pseudomanifold* (see VIII, Def. 3.11 and XI, 4.81).

In fact, since $K$ is a pure $n$-complex, it is easy to see that $O_Ke$ is also a pure $n$-complex.

If $T^{n-1}$ is an $(n - 1)$-simplex of $O_Ke$, $O_Ke$ contains the two $n$-simplexes incident to $T^{n-1}$ in $K$.

We shall now show that $O_Ke$ is a strongly connected complex. If this is not the case, then $O_Ke$ contains two $n$-simplexes $T_1{}^n$ and $T_{\mu+1}{}^n$ which cannot be joined by a chain of $n$-simplexes. Denote by $T_1{}^n, \cdots, T_\mu{}^n$ those $n$-simplexes of $O_Ke$ which can be connected with $T_1{}^n$ by chains of $n$-simplexes and let $T_{\mu+1}{}^n, \cdots, T_{\mu+\nu}{}^n$ be the remaining $n$-simplexes of $O_Ke$. Let $K'$ be the complex consisting of the first kind of $n$-simplexes and those of their faces which belong to $O_Ke$ and let $K''$ be the similarly defined complex for the second kind of $n$-simplexes. Then $K'$ and $K''$ are closed in $O_Ke$, $O_Ke = K' \cup K''$, and $\dim (K' \cap K'') \leq n - 2$. Since $e$ is a vertex of all the elements of $O_Ke$, and hence is contained in every nonempty closed subcomplex of $O_Ke$, it follows that $K' \cap K''$ is nonempty.

According to the definition of $K'$ and $K''$ it follows further that if $T^{n-1}$ is an $(n - 1)$-simplex of either complex, then both $n$-simplexes incident with $T^{n-1}$ belong to the same complex, $K'$ or $K''$, which contains $T^{n-1}$. Therefore, assigning to every $n$-simplex of $K'$ the coefficient 1, we obtain an $n$-cycle (mod 2):

$$z_1{}^n \in Z_\Delta{}^n(K', J_2) \subseteq Z_\Delta{}^n(O_Ke, J_2).$$

In the same way, the analogous $n$-chain in $K''$ is a cycle (mod 2):

$$z_2{}^n \in Z_\Delta{}^n(K'', J_2) \subseteq Z_\Delta{}^n(O_Ke, J_2).$$

Since the cycles $z_1{}^n$ and $z_2{}^n$ are not zero and are both in $Z_\Delta{}^n(O_Ke, J_2)$, the order of $Z_\Delta{}^n(O_Ke, J_2) = \Delta^n(O_Ke, J_2)$ is greater than 2. Since $\Delta_0{}^n(O_Ke)$ is infinite cyclic and $\Delta_0{}^{n-1}(O_Ke) = 0$, it follows that (IX, Theorem 4.1) $\Delta^n(O_Ke, J_2)$ is of order 2. This contradiction shows that $O_Ke$ is strongly connected; hence it is an $n$-dimensional pseudomanifold.

Since $\Delta_0{}^n(O_Ke) \neq 0$, $O_Ke$ is an orientable pseudomanifold. This proves 1.21.

THEOREM 1.22. *An h-manifold is a closed pseudomanifold.*

*Proof.* Let $K$ be a triangulation of an $n$-dimensional $h$-manifold. In view of 1.1, Remark 2, we need merely show that $K$ is a strongly connected complex. Suppose that this is not so. Then $K$ is a union of two nonempty pure $n$-dimensional closed subcomplexes $K'$ and $K''$ whose intersection is a complex of dimension $\leq n - 2$. Let $e$ be a vertex of $K' \cap K''$. Then

$$O_Ke = (O_Ke \cap K') \cup (O_Ke \cap K'').$$

Since $e \in K' \cap K''$ and $K'$, $K''$ are pure complexes, it follows that $O_K e \cap K'$ and $O_K e \cap K''$ are nonempty pure complexes. They are closed in $O_K e$ since $K'$, $K''$ are unrestricted complexes. Finally,

$$(O_K e \cap K') \cap (O_K e \cap K'') \subseteq K' \cap K'';$$

hence the dimension of $(O_K e \cap K') \cap (O_K e \cap K'')$ does not exceed $n - 2$. Consequently no $n$-simplex of $O_K e \cap K'$ can be connected to an $n$-simplex of $O_K e \cap K''$ by a chain of $n$-simplexes. Therefore, $O_K e$ is not strongly connected and so it cannot be a pseudomanifold. This contradicts Theorem 1.21 and proves Theorem 1.22.

Theorem 1.22 justifies

Definition 1.23. An $h$-manifold is said to be *orientable* if it is an orientable pseudomanifold. In the contrary case it is called *nonorientable.*

1.24. *In the notation of* 1.21 *the boundary* $B_K e$ *of the star* $O_K e$ *is an* $(n - 1)$-*dimensional simple* $h$-*manifold for* $n \geq 2$. [A pseudomanifold (in particular, an $h$-manifold) of dimension $n$ is said to be *simple* if it is orientable and its $\Delta^r$-groups are null groups for $0 < r < n$.]

*Proof.* By VIII, Corollary to Theorem 1.55, $\Delta_0^{r+1}(O_K e)$ is isomorphic to $\Delta_0^r(B_K e)$ for $r > 0$. This fact and 1.111 imply that $\Delta_0^r(B_K e) = 0$ $(0 < r < n - 1)$ and that $\Delta_0^{n-1}(B_K e)$ is infinite cyclic.

It remains to be shown that the groups $\Delta_0^r$ of the star of an arbitrary element $T$ of $B_K e$ relative to $B_K e$ are null for $0 < r < n - 1$ and that the group $\Delta_0^{n-1}$ of this star is infinite cyclic. Both assertions follow on setting $B_K e = B$ in IV, Theorem 2.31. Then $O_K eT = eO_B T$ and by VIII, Theorem 1.550, the groups $\Delta_0^r(O_B T)$, $\Delta_0^{r+1}(O_K eT)$ are isomorphic. (This proof of Theorem 1.24 is due to L. S. Pontryagin.)

**§1.3. The case** $n \leq 3$. Every one-dimensional closed pseudomanifold is homeomorphic to a circumference (VIII, 3.1, Example 1°); since a circumference is a manifold (hence certainly an $h$-manifold), it follows that the concepts of closed pseudomanifold, $h$-manifold and manifold coincide for $n = 1$.

Turning to the case $n = 2$, we note that there are closed orientable pseudomanifolds which are not $h$-manifolds: for example, a manifold of this sort is obtained by identifying two distinct points of a 2-sphere (see Fig. 122).

We know, further, that there exists an infinite set of closed 2-manifolds (closed surfaces) no two of which are homeomorphic, and that this set contains an infinite number of orientable and an infinite number of nonorientable manifolds. We shall prove that the set of closed surfaces exhausts the class of two-dimensional $h$-manifolds.

Indeed, let $K$ be a triangulation of a two-dimensional $h$-manifold $\|K\|$ and let $e$ be a vertex of $K$; then Theorem 1.24 implies that the boundary of the star $O_{Ke}$ is a closed one-dimensional combinatorial pseudomanifold, i.e., a closed polygon without multiple points. Hence the body of $O_{Ke}$ is homeomorphic to the interior of a circle. Thus, every two-dimensional $h$-manifold is a manifold.

We prove, finally, that every three-dimensional $h$-manifold $\| K \|$ is also a manifold. To this end it is sufficient to show that the body of an arbitrary star $O_{Ke}$ (notation as above) of $K$ is homeomorphic to the interior of a three-dimensional solid sphere (an open 3-cell). This will follow if we show that the body of the boundary $B_{Ke}$ of $O_{Ke}$ is homeomorphic to a 2-sphere. But, according to 1.24, the complex $B_{Ke}$ is a simple two-dimensional $h$-manifold, that is, the body of $B_{Ke}$ is a closed surface whose one-dimensional $\Delta$-group is the null group; this means that the surface is homeomorphic to a 2-sphere, and this is what we were to show.

We have proved the following proposition:

1.3. *Every $n$-dimensional closed $h$-manifold is a manifold for $n \leq 3$.*

We note that this is no longer true for $n \geq 4$; for each of these values of $n$ there are closed $h$-manifolds *which are not manifolds.* The fact is that for arbitrary $n \geq 4$ there are closed $(n - 1)$-manifolds (the so called Poincaré spaces) which are not homeomorphic to the $(n - 1)$-sphere but whose $\Delta$-groups, in spite of this, are isomorphic to the $\Delta$-groups of the $(n - 1)$-sphere. Assuming that this fact has been proved, we take in $R^{2n-1}$ a polyhedral complex $K$, which is a triangulation of an $(n - 1)$-dimensional Poincaré space; we assume also that all the vertices of $K$ are in general position. Choose two points $e_0$ and $e_1$ in $R^{2n+1}$ so that the set consisting of these two points and all the vertices of $K$ is in general position. If each point of $K$ is connected to both points $e_0$, $e_1$ by straight line segments, we obtain two pyramids with common base $K$ and vertices $e_0$, $e_1$. The union of the two pyramids is an $n$-complex $K''$; it is easy to see that $K''$ is a triangulation of an $n$-dimensional $h$-manifold. However, $K''$ is not a manifold, since $O_{K}e_0$ and $O_{K}e_1$ are not homeomorphic to the interior of a solid $n$-sphere (an open $n$-cell).

Hence the construction of $h$-manifolds which are not manifolds is reduced to the construction of Poincaré spaces.

The following example, taken from Seifert-Threlfall (see vol. 1, Bibliography, [S-T; p. 216]), is apparently the simplest example of a three-dimensional Poincaré space. The required manifold is obtained by identifying opposite faces of a dodecahedron, first twisting one of each pair of opposite faces through an angle $\pi/5$ relative to the other. The exact scheme of this identification is shown in Fig. 138, which shows 11 faces of the dodecahedron; the twelfth face is the exterior of the figure. The identification of the elements (vertices, edges, and faces) is indicated in the figure by the assignment of the same letters or numbers to the elements which are to be

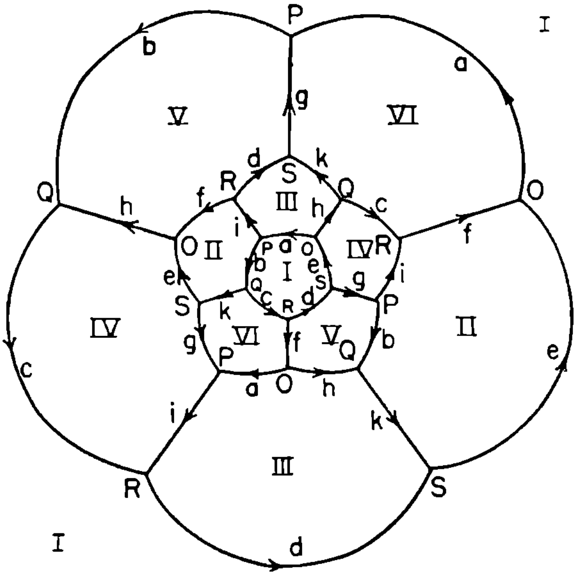


FIG. 138

identified. In particular, the twelfth face is denoted by $I$. For a proof of the fact that the result of the identification is indeed a three-dimensional Poincaré space the reader is referred to Seifert-Threlfall ([S-T; p. 216 ff.]).

**§1.4. Barycentric stars and $h$-manifolds.** Let $K$ be an $n$-dimensional $h$-manifold, and let $K_1'$ be the barycentric subdivision of the complex $K$. The following proposition is of fundamental importance in the theory of $h$-manifolds:

1.4. *All the barycentric stars of the complex $K$ are simple pseudomanifolds.*

*Proof.* The theorem is obvious for $n = 1$; assuming that it is true for $n - 1$, we prove it for $n$.

To this end, let $e$ be an arbitrary vertex of $K$ and assign to the centroid of each simplex $T^r \in O_K e$, $r > 0$, the centroid of the face $T^{r-1}$ of $T^r$ opposite the vertex $e$.

It is easy to see that this assignment is a (1—1) simplicial mapping (isomorphism) $S$ of the outer boundary $B^*$ of the barycentric star $T^{*n}(e)$ dual to $e$ onto the barycentric subdivision $(B_K e)_1$ of the complex $B_K e$ (Fig. 139). This mapping assigns to every simplex of $B^*$ with first vertex the centroid of some $T^r = eT^{r-1} \in O_K e$, $r > 0$, the simplex of $(B_K e)_1$ with first vertex the centroid of $T^{r-1} \in B_K e$. Hence, it follows easily that *every barycentric star of $K$ contained in $B^*$* (i.e., every element of the barycentric complex dual to the triangulation $K$ which is a subcomplex of the complex $B^*$) *is isomorphic to a barycentric star of $B_K e$* (that is, to a barycentric star which is an element of the barycentric complex dual to the triangulation $B_K e$).

But $B_K e$, by 1.24, is an $(n - 1)$-dimensional $h$-manifold. Hence, by the

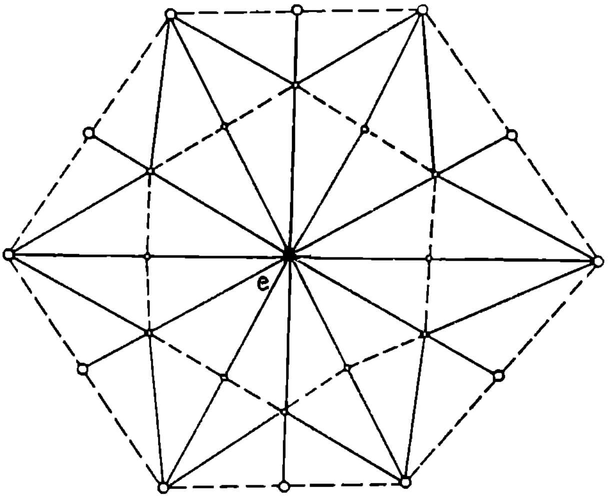


FIG. 139

inductive hypothesis, all the barycentric stars of $B_Ke$ are simple pseudomanifolds; consequently, all the barycentric stars of $K$ on $B^*$ are simple pseudomanifolds. Since every barycentric star of $K$ of dimension $\leq n - 1$ is on the outer boundary $B^*$ of some $n$-dimensional barycentric star $T^{*n}(e)$, we have proved that *every barycentric star of* $K$ *of dimension* $\leq n - 1$ *is a simple pseudomanifold.*

We need merely prove now that every $n$-dimensional barycentric star $T^{*n}(e)$, i.e., every star dual to a vertex $e$ of $K$, is a simple pseudomanifold. But this is an immediate consequence of 1.24 and of VIII, Corollary to Theorem 1.55 (also of 1.11 and the fact that $T^{*n}(e)$ is the star of $e$ in the complex $K_1$). This completes the proof of Theorem 1.4.

REMARK. We have proved incidentally that the complex $B^*$ [the outer boundary of the barycentric star $T^{*n}(e)$] is isomorphic to $(B_Ke)_1$ (the barycentric subdivision of the outer boundary $B_Ke$ of the star $O_Ke$).

## §2. The barycentric complex of a combinatorial $h$-manifold

**§2.1. Notation; preliminary basic facts.** Let $K$ be a triangulation of an $n$-dimensional $h$-manifold $\| K \|$. We denote by $K_1$ the barycentric subdivision of $K$ and by $T_i^p$ the $p$-simplexes of $K$; $*T_{1i}^q$ $(p + q = n)$ will stand for the barycentric stars dual to the simplexes $T_i^p$. According to 1.4 and IV, 5.24, *the barycentric stars of* $K$ *form a fundamental system of subcomplexes of the complex* $K_1$ (X, Def. 2.21). The same theorems imply a stronger form of this result, which we formulate below.

Let $K^*$ be the complex of all the barycentric stars of $K$ and let $K_0^*$ be a closed $m$-dimensional $(m \leq n)$ subcomplex of $K^*$. Denote by $K_{01} \subseteq K_1$ the barycentric subdivision of $K_0^*$ (see IV, 5.4), that is, the union of all the barycentric stars (as subcomplexes of $K_1$) which are elements of $K_0^*$;

we know (IV, 5.4) that $K_{01}$ is a triangulation. With these conditions, IV, Theorem 5.24 and Theorem 1.4 of this chapter imply that the elements of $K_0^*$ form a fundamental system of subcomplexes of the complex $K_{01}$.

From this and X, Theorem 2.32 it follows further that:

2.10. *If* $*t_{1i}^r$ *are arbitrary orientations of the stars* $*T_{1i}^r \in K_0^*$, *then the* $*t_{1i}^r$ *form a fundamental system of chains* (X, Def. 2.31) *of the complex* $K_{01}$.

If $K_0^* = K^*$, this theorem becomes

2.1. *The system of all oriented barycentric stars* $*t_{1i}^r$ *of the complex* $K$ *is a fundamental system of chains of the complex* $K_1$.

We formulate separately the propositions expressing the fact that the fundamental system of chains $*t_{1i}^r$ satisfies conditions 3° and 4° of X, Def. 2.31:

2.11. *Every homology class*

$$\mathfrak{z}^r \in \Delta^r(K_{01})$$

*contains the barycentric subdivision* $s_1 z^*$ *of a cycle* $z^* = \sum a_i {}^*t_{1i}^r$.

2.12. *If a cycle* $z_1^r = s_1 z^*$, *where* $z^* = \sum a_i {}^*t_{1i}^r$, *is homologous to zero in* $K_1$, *it is the boundary of a chain* ${}^*x^{r+1} = \sum e_i {}^*t_{1i}^{r+1}$.

§**2.2. The complex** $\mathfrak{K}^*$. We shall now construct, by the rule of X, 2.4, the $a$-complex $\mathfrak{K}^*$, defined by the fundamental system of chains $*t_{1i}^r$.

The elements of the complex $\mathfrak{K}^*$ will be denoted by $\pm {}^*t_i^r$, where ${}^*t_i^r$ corresponds to ${}^*t_{1i}^r$ and $-{}^*t_i^r$ to $-{}^*t_{1i}^r$. Since the pair of chains ${}^*t_{1i}^r$ and $-{}^*t_{1i}^r$ is completely defined by the orientable pseudomanifold ${}^*T_{1i}^r$ (as the pair of its orientations), the $a$-complex $\mathfrak{K}^*$ is completely defined by the complex $K^*$ in the same sense that a simplicial complex $K_0$ defines the $a$-complex of its oriented simplexes. Hence, extending this analogy with simplicial complexes, we will denote the groups

$$L^r(\mathfrak{K}^*), \qquad Z^r(\mathfrak{K}^*), \qquad \Delta^r(\mathfrak{K}^*), \qquad \nabla^r(\mathfrak{K}^*), \text{ etc.}$$

by

$$L^r(K^*), \qquad Z^r(K^*), \qquad \Delta^r(K^*), \qquad \nabla^r(K^*), \text{ etc.,}$$

respectively.

We may now state the following fundamental result on the basis of X, Theorem 2.5:

THEOREM 2.2. *The groups*

$$\Delta^r(K), \qquad \Delta^r(K_1), \qquad \Delta^r(K^*)$$

*are isomorphic; the groups*

$$\nabla^r(K), \qquad \nabla^r(K_1), \qquad \nabla^r(K^*)$$

*are also isomorphic.*

## §3. The intersection number, the isomorphism $D^q$ and the Poincaré duality

**§3.1. The intersection number ($t_1^p \times {}^*t_i^q$).** Let $\| K \|$ be an orientable $n$-dimensional $h$-manifold and choose a definite orientation of the combinatorial $h$-manifold $K$, that is, orientations $t_1^n, \cdots, t_{q(n)}^n$ of all its $n$-simplexes such that $\sum t_i^n$ is a cycle. Denote the induced orientations of the $n$-simplexes of $K_1$ by $t_{1\mu}^n$.

We consider an oriented simplex $t_i^p$ of $K$ and the barycentric star ${}^*T_{1i}^q$ $(p + q = n)$ dual to the simplex $| t_i^p |$, and give this star an arbitrary orientation ${}^*i_{1i}^q$. The assignment of the orientation ${}^*t_{1i}^q$ is equivalent to the assignment of the element ${}^*t_i^q$ of the cell complex $K^*$. Let $| t_{1\mu}^p |$ be a simplex of $K_1$ contained in $| t_i^p |$:

$$| t_{1\mu}^p | = (a_{1i(0)} a_{1i(1)} \cdots a_{1i(p-1)} a_{1i}),$$

where $a_{1i(0)} < a_{1i(1)} < \cdots < a_{1i}$ (see IV, 5.1, where the vertices are written in the reverse order), i.e., $a_{1i(0)}, a_{1i(1)}, \cdots, a_{1i}$ are the centroids of the faces

$$T_{i(0)}^0 < T_{i(1)}^1 < \cdots < T_{i(p-1)}^{p-1} < T_i^p$$

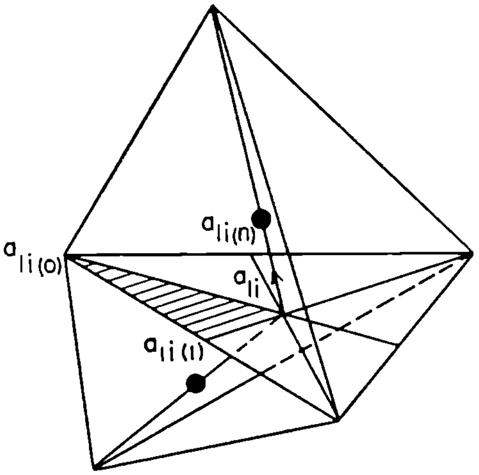
of the simplex $T_i^p$ (Fig. 140). On the other hand, let

$$| t_{1\nu}^q | = (a_{1i} a_{1i(p+1)} \cdots a_{1i(n)})$$

be a $q$-simplex of ${}^*T_{1i}^q$, with

$$a_{1i} < a_{1i(p+1)} < \cdots < a_{1i(n)},$$

i.e., $a_{1i}, a_{1i(p+1)}, \cdots, a_{1i(n)}$ are the centroids of the simplexes $T_i^p < T_{i(p+1)}^{p+1} < \cdots < T_{i(n)}^n$. We give the simplexes $| t_{1\mu}^p |$ and $| t_{1\nu}^q |$ orienta-



n=3, p=2, q=1

FIG. 140

tions coherent with the chosen orientations $t_i^p$ of $T_i^p$ and $*t_{1i}^q$ of $*T_{1i}^q$, respectively. Let these orientations be

$$\left.\begin{aligned} t_{1\mu}^p &= \varepsilon \mid a_{1i(0)} \cdots a_{1i(p-1)}a_{1i} \mid, \\ t_{1\nu}^q &= \eta \mid a_{1i(0)}a_{1i(p+1)} \cdots a_{1i(n)} \mid, \end{aligned}\right\} \qquad \varepsilon = \pm 1; \ \eta = \pm 1.$$

The simplex $(a_{1i(0)} \cdots a_{1i(p-1)}a_{1i}a_{1i(p+1)} \cdots a_{1i(n)})$ is a simplex $\mid t_{1\lambda}^n \mid$ of $K_1$ contained in $T_{i(n)}^n$, where $t_{1\lambda}^n$ denotes that orientation of the simplex $\mid t_{1\lambda}^n \mid$ induced by the orientation of the whole $h$-manifold $K$. Then

$$t_{1\lambda}^n = \gamma \mid a_{1i(0)} \cdots a_{1i(p-1)}a_{1i(p+1)} \cdots a_{1i(n)} \mid,$$

where $\gamma = \pm 1$.

We shall prove that the number $\varepsilon\eta\gamma = \pm 1$ is independent of the choice of $\mid t_{1\mu}^p \mid$ and $\mid t_{1\nu}^q \mid$, and consequently depends only on $t_i^p$, $*t_{1i}^q$ and the prescribed orientation of the $h$-manifold $K$.

The barycentric subdivision of the simplex $\mid t_i^p \mid$ consisting of the simplexes $\mid t_{1\mu}^p \mid$, $\mid t_{1\mu'}^p \mid$, $\cdots$ and their faces is a strongly connected complex. Therefore, if we prove that the number $\varepsilon\eta\gamma$ remains unchanged when the simplex $\mid t_{1\mu}^p \mid$ is replaced by a simplex $\mid t_{1\mu'}^p \mid$ having a common $(p - 1)$-face with $\mid t_{1\mu}^p \mid$, we shall at the same time have proved that $\varepsilon\eta\gamma$ is independent of the choice of $\mid t_{1\mu}^p \mid$.

Hence, suppose that $\mid t_{1\mu}^p \mid$ and $\mid t_{1\mu'}^p \mid$ have a common $(p - 1)$-face $(a_{1i(0)} \cdots a_{1i(k-1)}a_{1i(k+1)} \cdots a_{1i})$, so that

$$\begin{aligned} \mid t_{1\mu}^p \mid &= (a_{1i(0)} \cdots a_{1i(k)} \cdots a_{1i}), \\ \mid t_{1\mu'}^p \mid &= (a_{1i(0)} \cdots a_{1i(k')} \cdots a_{1i}). \end{aligned}$$

Then the simplexes

$$\mid t_{1\lambda}^n \mid = (a_{1i(0)} \cdots a_{1i(k)} \cdots a_{1i}a_{1i(p+1)} \cdots a_{1i(n)})$$

and

$$\mid t_{1\lambda'}^n \mid = (a_{1i(0)} \cdots a_{1i(k')} \cdots a_{1i}a_{1i(p+1)} \cdots a_{1i(n)})$$

have a common $(n - 1)$-face

$$(a_{1i(0)} \cdots a_{1i(k-1)}a_{1i(k+1)} \cdots a_{1i}a_{1i(p+1)} \cdots a_{1i(n)}).$$

If

$$t_{1\mu}^p = \varepsilon \mid a_{1i(0)} \cdots a_{1i(k)} \cdots a_{1i} \mid$$

and

$$t_{1\mu'}^p = \varepsilon' \mid a_{1i(0)} \cdots a_{1i(k')} \cdots a_{1i} \mid$$

and $t^{p-1}$ is any orientation of the simplex

$$(a_{1i(0)} \cdots a_{1i(k-1)}a_{1i(k+1)} \cdots a_{1i}),$$

e.g.,

$$t^{p-1} = |\, a_{1i(0)} \cdots a_{1i(k-1)} a_{1i(k+1)} \cdots a_{1i} \,|,$$

then

$$(t_{1\mu}^{p} : t^{p-1}) = -(t_{1\mu'}^{p} : t^{p-1}),$$

since $t_{1\mu}^p$ and $t_{1\mu'}^p$ are coherently oriented and are situated on different sides of their common face.

But

$$(t_{1\mu}^{p} : t^{p-1}) = (-1)^k \varepsilon,$$
$$(t_{1\mu'}^{p} : t^{p-1}) = (-1)^k \varepsilon'.$$

Hence $\varepsilon' = -\varepsilon$. Analogously, if the orientations

$$t_{1\lambda}^{n} = \gamma \,|\, a_{1i(0)} \cdots a_{1i(k)} \cdots a_{1i} a_{1i(p+1)} \cdots a_{1i(n)} \,|$$

and

$$t_{1\lambda'}^{n} = \gamma' \,|\, a_{1i(0)} \cdots a_{1i(k')} \cdots a_{1i} a_{1i(p+1)} \cdots a_{1i(n)} \,|$$

are equal to the orientation $t_{i(n)}^n$ and are therefore coherent, and

$$t^{n-1} = |\, a_{1i(0)} \cdots a_{1i(k-1)} a_{1i(k+1)} \cdots a_{1i} a_{1i(p+1)} \cdots a_{1i(n)} \,|,$$

then

$$(t_{1\lambda}^{n} : t^{n-1}) = -(t_{1\lambda'}^{n} : t^{n-1}),$$
$$(t_{1\lambda}^{n} : t^{n-1}) = (-1)^k \gamma,$$
$$(t_{1\lambda'}^{n} : t^{n-1}) = (-1)^k \gamma',$$
$$\gamma' = -\gamma.$$

Therefore,

$$\varepsilon' \eta \gamma' = (-\varepsilon)\eta(-\gamma) = \varepsilon \eta \gamma,$$

so that $\varepsilon\eta\gamma$ does not depend on the replacement of $|\, t_{1\mu}^p \,|$ by $|\, t_{1\mu'}^p \,|$. In exactly the same way, using the strong connectedness of the complex $*T_{1i}^q$ we prove that $\varepsilon\eta\gamma$ is independent of the choice of $|\, t_{1\nu}^q \,|$.

Definition 3.1. The number $\varepsilon\eta\gamma = \pm 1$, depending only on $t_i^p$, $*t_i^q$ and the prescribed orientation of the $h$-manifold $K$ is called the *intersection number* of $t_i^p$ and $*t_i^q$ in the oriented $h$-manifold $K^*$ and is denoted by $(t_i^p \times *t_i^q)$.

Remark 1. Let $R^n$, $R_{1\mu}^p$, $R_{1\nu}^q$ be the carrying planes of the simplexes $|\, t_{1\lambda}^n \,|$, $|\, t_{1\mu}^p \,|$, $|\, t_{1\nu}^q \,|$, respectively, oriented in the same way as $t_{1\lambda}^n$, $t_{1\mu}^p$, $t_{1\nu}^q$. Then $(t_i^p \times *t_i^q)$ is precisely the intersection number $(R_{1\mu}^p \times R_{1\nu}^q)$ defined in VII, 2.1.

The intersection number $(t_i^p \times {}^*t_h^q)$ is, by definition, set equal to zero for $h \neq i$ (that is, if the barycentric star ${}^*T_{1h}^q$ is not dual to the simplex $T_i^p$).

REMARK 2. If $t_i^p$ $({}^*t_i^q)$ is replaced by $-t_i^p$ $(-{}^*t_i^q)$, $\varepsilon$ $(\eta)$ changes to $-\varepsilon$ $(-\eta)$. Hence

$$((-t_i^p) \times {}^*t_i^q) = (t_i^p \times (-{}^*t_i^q)) = -(t_i^p \times {}^*t_i^q).$$

Furthermore, if the given orientation of the $h$-manifold $K$ is replaced by its opposite, $\gamma$ changes to $-\gamma$. Hence the intersection number of $t_i^p$ and ${}^*t_i^q$ changes sign if the orientation of the $h$-manifold is replaced by its negative.

### §3.2. The intersection number and the incidence numbers.

LEMMA 3.21. *If* $|t_j^{p-1}| < |t_i^p|$, *then*

$$(3.2) \qquad (t_i^p \times {}^*t_i^q) = (t_i^p : t_j^{p-1})({}^*t_j^{q+1} : {}^*t_i^q)(-1)^p(t_j^{p-1} \times {}^*t_j^{q+1}).$$

*Proof*. Since $|t_j^{p-1}| < |t_i^p|$ implies that $|{}^*t_i^q| < |{}^*t_j^{q+1}|$ and hence that $(t_i^p : t_j^{p-1}) = \pm 1$, $({}^*t_j^{q+1} : {}^*t_i^q) = \pm 1$, it follows that the right and left sides of (3.2) are both equal to 1 in absolute value. We need only show that both sides have the same sign.

We consider first the very important special case

$$(3.21) \qquad (t_i^p : t_j^{p-1}) = 1, \qquad ({}^*t_j^{q+1} : {}^*t_i^q) = 1.$$

Let

$$|t_{1\mu}^p| = (a_{1i(0)} \cdots a_{1i(p-2)}a_{1j}a_{1i})$$

be a simplex on (contained in) $|t_i^p|$, with $a_{1i(0)}, \cdots, a_{1i(p-2)}, a_{1j}, a_{1i}$ the centroids of $T_{i(0)}^0 < \cdots < T_{i(p-2)}^{p-2} < T_j^{p-1} < T_i^p$ (Fig. 141); and suppose

$$|t_{1\sigma}^{q+1}| = (a_{1j}a_{1i}a_{1i(p+1)} \cdots a_{1i(n)}) \in {}^*T_{1j}^{q+1}.$$

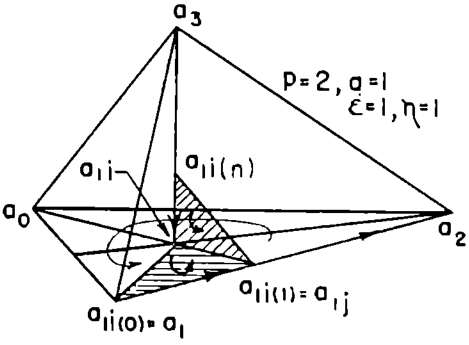


FIG. 141

The orientations $t_{1\mu}^{p}$ and $t_{1\sigma}^{q+1}$ of $|t_{1\mu}^{p}|$ and $|t_{1\sigma}^{q+1}|$ are taken to be coherent with the orientations $t_i^p$ and $*t_{1j}^{q+1}$:

$$t_{1\mu}^{p} = \varepsilon\, |a_{1i(0)} \cdots a_{1j}a_{1i}|, \qquad t_{1\sigma}^{q+1} = \eta\, |a_{1j}a_{1i}a_{1i(p+1)} \cdots a_{1i(n)}|.$$

Then $|t_{1\rho}^{p-1}| = (a_{1i(0)} \cdots a_{1j})$ is on $|t_j^{p-1}| < |t_i^p|$, so that if $t_{1\rho}^{p-1} = \varepsilon\, |a_{1i(0)} \cdots a_{1j}|$, we have

$$(t_{1\mu}^{p} : t_{1\rho}^{p-1}) = (-1)^p.$$

Hence, if $s_1 t_i^p$ is the barycentric subdivision of the chain $t_i^p$:

$$s_1 t_i^p = t_{1\mu}^p + \cdots,$$

we get

$$\Delta s_1 t_i^p = \Delta t_{1\mu}^p + \cdots = (-1)^p t_{1\rho}^p + \cdots. \tag{3.210}$$

But

$$\Delta s_1 t_i^p = s_1 \Delta t_i^p,$$

and according to (3.21),

$$\Delta t_i^p = t_j^{p-1} + \cdots;$$

therefore

$$s_1 \Delta t_i^p = s_1 t_j^{p-1} + \cdots,$$

that is,

$$\Delta s_1 t_i^p = s_1 t_j^{p-1} + \cdots. \tag{3.211}$$

Comparing (3.210) with (3.211), we obtain

$$s_1 t_j^{p-1} = (-1)^p t_{1\rho}^{p-1} + \cdots.$$

It follows that *the orientation* $(-1)^p t_{1\rho}^{p-1}$ *of the simplex* $|t_{1\rho}^{p-1}|$ *is coherent with the orientation* $t_j^{p-1}$ *of its carrier* $|t_j^{p-1}|$.

Similarly,

$$|t_{1\nu}^q| = (a_{1i}a_{1i(p+1)} \cdots a_{1i(n)}) \in *T_{1i}^q < *T_{1j}^{q+1},$$

so that if $t_{1\nu}^q = \eta\, |a_{1i}a_{1i(p+1)} \cdots a_{1i(n)}|$,

$$(t_{1\sigma}^{q+1} : t_{1\nu}^q) = 1.$$

But

$$*t_{1j}^{q+1} = t_{1\sigma}^{q+1} + \cdots,$$

$$\Delta *t_{1j}^{q+1} = \Delta t_{1\sigma}^{q+1} + \cdots = t_{1\nu}^q + \cdots.$$

On the other hand, according to (3.21),

$$\Delta {}^*t_{1j}^{q+1} = {}^*t_{1i}^{q} + \cdots ;$$

whence it follows that

$${}^*t_{1i}^{q} = t_{1\nu}^{q} + \cdots ,$$

that is, $t_{1\nu}^q$ *is coherent with the orientation* ${}^*t_{1i}^q$ *of the pseudomanifold* ${}^*T_{1i}^q$.

Finally, suppose that the prescribed orientation of the pseudomanifold $K_1$ is defined by the oriented simplex

$$t_{1\lambda}^{n} = \gamma \mid a_{1i(0)} \cdots a_{1i(p-2)} a_{1j} a_{1i} a_{1i(p+1)} \cdots a_{1i(n)} \mid .$$

To summarize: the orientation

$$t_{1\mu}^{p} = \varepsilon \mid a_{1i(0)} \cdots a_{1i(p-2)} a_{1j} a_{1i} \mid$$

is coherent with $t_i^p$; the orientation

$$t_{1\nu}^{q} = \eta \mid a_{1i} a_{1i(p+1)} \cdots a_{1i(n)} \mid$$

is coherent with ${}^*t_{1i}^q$; the orientation

$$t_{1\lambda}^{n} = \gamma \mid a_{1i(0)} \cdots a_{1i(p-2)} a_{1j} a_{1i} a_{1i(p+1)} \cdots a_{1i(n)} \mid$$

is coherent with the orientation of $K_1$. Therefore, for a given orientation of $K_1$

$$(t_i^p \times {}^*t_i^q) = \varepsilon\eta\gamma. \tag{3.221}$$

On the other hand, the orientation $(-1)^p t_{1\rho}^{p-1} = (-1)^p \varepsilon \mid a_{1i(0)} \cdots a_{1j} \mid$ is coherent with $t_j^{p-1}$, the orientation $t_{1\sigma}^{q+1} = \eta \mid a_{1j} a_{1i} a_{1i(p+1)} \cdots a_{1i(n)} \mid$ is coherent with ${}^*t_{1j}^{q+1}$ and, as before, the orientation

$$\gamma \mid a_{1i(0)} \cdots a_{1i(p-2)} a_{1j} a_{1i(p+1)} \cdots a_{1i(n)} \mid$$

is coherent with the orientation of $K_1$. Then, for the given orientation of $K_1$,

$$(t_j^{p-1} \times {}^*t_j^{q+1}) = (-1)^p \varepsilon\eta\gamma, \tag{3.23}$$

i.e.,

$$(t_i^p \times {}^*t_i^q) = (-1)^p (t_j^{p-1} \times {}^*t_j^{q+1}).$$

Hence (3.2) is proved on the assumption that (3.21) holds.

Assume now that (3.21) does not hold and let

$$(t_i^p : t_j^{p-1}) = \varepsilon_{ij}, \qquad ({}^*t_j^{q+1} : {}^*t_i^q) = \eta_{ji}.$$

Then

$$(\varepsilon_{ij} t_i^p : t_j^{p-1}) = \varepsilon_{ij}^2 = 1; \qquad (\eta_{ji} {}^*t_j^{q+1} : {}^*t_i^q) = \eta_{ji}^2 = 1,$$

and, consequently, in view of what we have shown,

$$(\varepsilon_{ij}t_i^p \times {}^*t_i^q) = (-1)^p(t_j^{p-1} \times \eta_{ji}{}^*t_j^{q+1}).$$

But

$$(t_i^p \times {}^*t_i^q) = \varepsilon_{ij}^2(t_i^p \times {}^*t_i^q) = \varepsilon_{ij}(\varepsilon_{ij}t_i^p \times {}^*t_i^q),$$

$$(t_j^{p-1} \times \eta_{ji}{}^*t_j^{q+1}) = \eta_{ji}(t_j^{p-1} \times {}^*t_j^{q+1}).$$

Therefore,

$$(t_i^p \times {}^*t_i^q) = \varepsilon_{ij}(\varepsilon_{ij}t_i^p \times {}^*t_i^q) = \varepsilon_{ij}(-1)^p(t_j^{p-1} \times \eta_{ji}{}^*t_j^{q+1})$$

$$= \varepsilon_{ij}\eta_{ji}(-1)^p(t_j^{p-1} \times {}^*t_j^{q+1}).$$

This completes the proof of Lemma 3.21.

**§3.3. The isomorphism $D^q$ and the Poincaré duality.** Let $t_i^p$ be an arbitrary but fixed orientation of the simplex $T_i^p$ of an orientable (and oriented) $n$-dimensional $h$-manifold $K$. Of the two possible orientations of the orientable pseudomanifold ${}^*T_{1i}^q$ denote by ${}^*t_{1i}^q$ the one for which

$$(t_i^p \times {}^*t_i^q) = 1.$$

We make this assumption for each dimension $p$ and examine the resulting incidence matrices $\| ({}^*t_j^{q+1} : {}^*t_i^q) \| = \| \eta_{ji}^q \|$.

First, if $| t_j^{p-1} |$ is not a face of $| t_i^p |$, then $(t_i^p : t_j^{p-1}) = 0$ and $({}^*t_j^{q+1} : {}^*t_i^q) = 0$.

If $| t_j^{p-1} | < | t_i^p |$, i.e., if $(t_i^p : t_j^{p-1}) \neq 0$, we have by Lemma 3.21,

$$(t_i^p \times {}^*t_i^q) = (-1)^p(t_i^p : t_j^{p-1})({}^*t_j^{q+1} : {}^*t_i^q)(t_j^{p-1} \times {}^*t_j^{q+1});$$

or, recalling that $(t_i^p \times {}^*t_i^q) = 1$, $(t_j^{p-1} \times {}^*t_j^{q+1}) = 1$,

$$1 = (-1)^p(t_i^p : t_j^{p-1})({}^*t_j^{q+1} : {}^*t_i^q).$$

Therefore, multiplying both sides by $(-1)^p(t_i^p : t_j^{p-1})$ and remembering that

$$(t_i^p : t_j^{p-1}) = \pm 1,$$

we obtain

$$({}^*t_j^{q+1} : {}^*t_i^q) = (-1)^p(t_i^p : t_j^{p-1}). \tag{3.31}$$

We now assign to each chain $x^p \in L^p(K, \mathfrak{A})$ the chain ${}^*x^q \in L^q(K^*, \mathfrak{A})$ which assumes on each ${}^*t_i^q$ the same value that $x^p$ assumes on $t_i^p$:

$$({}^*x^q \cdot {}^*t_i^q) = (x^p \cdot t_i^p).$$

We denote the chain ${}^*x^q$ by $D^q x^p$. *The mapping $D^q$ is obviously an isomorphism of $L^p(K)$ onto $L^q(K^*)$.*

We shall prove the following fundamental property of $D^q$:

$$\Delta D^q x^p = (-1)^{p+1} D^{q-1} \nabla x^p, \qquad (3.32)$$
$$\nabla D^q x^p = (-1)^p D^{q+1} \Delta x^p.$$

To prove this, we set

$$\varepsilon_{ij}^{p-1} = (t_i^p : t_j^{p-1}), \qquad \eta_{ji}^{q+1} = ({}^*t_j^{q+1} : {}^*t_i^q)$$

for brevity and recall that by (3.31)

$$\varepsilon_{ij}^{p-1} = (-1)^p \eta_{ji}^{q+1},$$

so that

$$\varepsilon_{ij}^p = (-1)^{p+1} \eta_{ji}^q.$$

Then

$$(\Delta D^q x^p \cdot {}^*t_j^{q-1}) = \textstyle\sum_i \eta_{ij}^q (D^q x^p \cdot {}^*t_i^q) = \sum_i \eta_{ij}^q (x^p \cdot t_i^p)$$
$$= (-)^{p+1} \textstyle\sum_i \varepsilon_{ji}^p (x^p \cdot t_i^p)$$
$$= (-1)^{p+1} (\nabla x^p \cdot t_j^{p+1}) = (-1)^{p+1} (D^{q-1} \nabla x^p \cdot {}^*t_j^{q-1}).$$

Similarly,

$$(\nabla D^q x^p \cdot {}^*t_j^{q+1}) = \textstyle\sum_i \eta_{ji}^{q+1} (D^q x^p \cdot {}^*t_i^q) = \sum_i \eta_{ji}^{q+1} (x^p \cdot t_i^p)$$
$$= (-1)^p \textstyle\sum \varepsilon_{ij}^{p-1} (x^p \cdot t_i^p) = (-1)^p (\Delta x^p \cdot t_j^{p-1})$$
$$= (-1)^p (D^{q+1} \Delta x^p \cdot {}^*t_j^{q+1}).$$

REMARK 1. We rewrite the identities we have just proved:

$$(\Delta D^q x^p \cdot {}^*t_j^{q-1}) = (-1)^{p+1} (\nabla x^p \cdot t_j^{p+1}), \qquad (3.320)$$
$$(\nabla D^q x^p \cdot {}^*t_j^{q+1}) = (-1)^p (\Delta x^p \cdot t_i^{p-1}).$$

The first equality of (3.32) implies:

a) If $\nabla x^p = 0$, then $\Delta D^q x^p = 0$, i.e., $D^q$ maps $Z_\nabla{}^p(K)$ into $Z_\Delta{}^q(K^*)$.

b) If $\Delta D^q x^p = 0$, then $D^{q-1} \nabla x^p = 0$, so that $\nabla x^p = 0$; in other words: if the image of a chain $x^p \in L^p(K)$ under $D^q$ is a cycle, then $x^p$ is a cocycle, or: the inverse image of $Z_\Delta{}^q(K^*)$ under the isomorphism $D^q$ of $L^p(K)$ onto $L^q(K^*)$ is $Z_\nabla{}^p(K)$.

From a) and b) we conclude that:

3.321$_\nabla$. *The isomorphism $D^q$ of $L^p(K)$ onto $L^q(K^*)$ maps $Z_\nabla{}^p(K)$ onto $Z_\Delta{}^q(K^*)$.*

Similarly, we deduce from the second equality of (3.32):

3.321$_\Delta$. *The isomorphism $D^q$ of $L^p(K)$ onto $L^q(K^*)$ maps $Z_\Delta{}^p(K)$ onto $Z_\nabla{}^q(K^*)$.*

We shall prove, moreover:

3.322$_\nabla$. *The isomorphism* $D^q$ *of* $L^p(K)$ *onto* $L^q(K^*)$ *maps* $H_\nabla{}^p(K)$ *onto* $H_\Delta{}^q(K^*)$.

*Proof.*

a) If $z^p \in H_\nabla{}^p(K)$, there is a chain $x^{p-1} \in L^{p-1}(K)$ such that

$$\nabla x^{p-1} = z^p.$$

Then, by the first formula of (3.32),

$$\Delta D^{q+1}x^{p-1} = (-1)^p D^q \nabla x^{p-1} = (-1)^p D^q z^p,$$

that is,

$$D^q z^p \in H_\Delta{}^q(K^*).$$

b) If $D^q z^p \in H_\Delta{}^q(K^*)$, there is a chain ${}^*x^{q+1} \in L^{q+1}(K^*)$ such that

$$\Delta {}^*x^{q+1} = D^q z^p.$$

Denote the isomorphism of $L^q(K^*)$ onto $L^p(K)$ inverse to $D^q$ by $D_p$ and put

$$x^{p-1} = D_{p-1}{}^*x^{q+1}.$$

Then [reading the first formula of (3.32) from right to left and replacing $p$ by $p - 1$] we obtain

$$(-1)^p D^q \nabla x^{p-1} = \Delta D^{q+1} x^{p-1} = \Delta D^{q+1} D_{p-1}{}^*x^{q+1} = \Delta {}^*x^{q+1} = D^q z^p;$$

whence [operating on $(-1)^p D^q \nabla x^{p-1}$ and $D^q z^p$ with the operator $D_p$],

$$(-1)^p \nabla x^{p-1} = z^p,$$

that is,

$$z^p \in H_\nabla{}^p(K).$$

This proves Theorem 3.322$_\nabla$.

Similarly, using the second formula of (3.32), we obtain

3.322$_\Delta$. *The isomorphism* $D^q$ *maps* $H_\Delta{}^p(K)$ *onto* $H_\nabla{}^q(K^*)$.

3.321$_\nabla$ and 3.322$_\nabla$ yield

3.32$_\nabla$. *The isomorphism* $D^q$ *of* $L^p(K)$ *onto* $L^q(K^*)$ *induces an isomorphism of* $\nabla^p(K)$ *onto* $\Delta^q(K^*)$.

Since the groups $\Delta^q(K^*)$ and $\Delta^q(K)$ are isomorphic (Theorem 2.2), we obtain

THEOREM 3.33. THE POINCARÉ DUALITY (KOLMOGOROV'S FORMULATION). *If* $\| K \|$ *is an orientable n-dimensional h-manifold and p, q are two arbitrary nonnegative integers whose sum is n, then the groups* $\nabla^p(\| K \|)$ *and* $\Delta^q(\| K \|)$ *are isomorphic for any coefficient domain.*

*Special Case.* Let $\mathfrak{A} = J$. Since (see IX, Theorem 3.51 and VIII, 2.21)

$$\nabla_0^p(\| K \|) = \Theta^{p-1}(\| K \|) + \Delta_{00}^p(\| K \|),$$

$$\Delta_0^q(\| K \|) = \Theta^q(\| K \|) + \Delta_{00}^q(\| K \|),$$

we have

3.330. *The torsion groups* $\Theta^{p-1}(\| K \|)$ *and* $\Theta^q(\| K \|)$ *are isomorphic and the Betti numbers* $\pi^p(\| K \|)$, $\pi^q(\| K \|)$ *are equal.*

Theorem 3.330 is the *Poincaré duality in its original form.*

Setting $\mathfrak{A} = J_m$, we deduce the isomorphism of the groups $\nabla_m^p(\| K \|)$ and $\Delta_m^q(\| K \|)$ from Theorem 3.33; hence, because of the isomorphism of $\nabla_m^p(\| K \|)$ and $\Delta_m^p(\| K \|)$ [IX, (4.410)] we have

3.33$_m$. The Poincaré Duality (mod $m$). *The groups* $\Delta_m^p(\| K \|)$ *and* $\Delta_m^q(\| K \|)$ *are isomorphic for every* $m$.

All these results hold for *orientable* $n$-dimensional $h$-manifolds; for a nonorientable $n$-dimensional $h$-manifold $\| K \|$, $\pi^0(\| K \|) = 1$ and $\pi^n(\| K \|) = 0$.

It is, therefore, all the more important to emphasize

3.33$_2$. The Poincaré Duality (mod 2). *If* $\| K \|$ *is an $n$-dimensional $h$-manifold (not necessarily orientable), then the groups* $\Delta_2^p(\| K \|)$ *and* $\Delta_2^q(\| K \|)$ *are isomorphic.*

To prove this theorem, it is enough to introduce (mod 2) terminology in the above arguments wherever intersections occur, that is, in all the calculations of this section integers should be replaced by their residues (mod 2) and all simplexes and stars should be taken as nonoriented. As a result, all the calculations are greatly simplified, since signs need no longer be taken into account, and become applicable to all (and not just to orientable) $h$-manifolds.

Theorem 3.33$_2$ implies that $\pi_2^p(\| K \|) = \pi_2^q(\| K \|)$ and this, for odd $n$, yields the formula

$$\sum_{r=0}^{n} (-1)^r \rho^r(K) = \sum_{r=0}^{n} (-1)^r \pi_2^r(K) = 0.$$

Therefore,

3.34. *The Euler characteristic of an arbitrary $h$-manifold of odd dimension is equal to zero.*

Remark 2. In particular, triangulations of three-dimensional manifolds have zero characteristic. A special case of this is the following proposition, whose proof is not difficult (for a proof see Seifert-Threlfall [S-T; Chapter 9, §60]; see Bibliography, vol. 1):

Suppose that $K_\alpha^3$ is a triangulation of a convex polyhedral domain $\| K_\alpha^3 \|$ and that the triangulation $K_\beta^3$ is the result of identifying certain elements of $K_\alpha^3$ which lie on the boundary of $\| K_\alpha^3 \|$. The complex $K_\beta^3$ is a triangulation of a three-dimensional manifold if, and only if, its Euler characteristic is zero.

REMARK 3. Putting $p = n$ in 3.330, so that $q = 0$, we see that $\Theta^{n-1}(\| K \|)$ is isomorphic to $\Theta^0(\| K \|)$ and is consequently the null group. Hence a special case of the Poincaré duality is a proposition already known to us (IX, Theorem 3.52): *An orientable n-dimensional h-manifold is* $(n - 1)$-*torsion free.*

**§3.4. The intersection number $(x^p \times *x^q)$, where $x^p \in L^p(K, \mathfrak{A})$, $*x^q \in L^q(K^*, \mathfrak{A})$ and $\mathfrak{A}$ is a ring.** Retaining all notation introduced at the beginning of this section, we choose the orientations $t_i^p$ and $*t_i^q$ so that $(t_i^p \times *t_i^q) = 1$; hence for arbitrary $i$ and $j$,

$$(t_i^p \times *t_j^q) = \delta_{ij}.$$

Let

$$x^p = \sum a_i t_i^p \in L^p(K, \mathfrak{A}), \qquad *x^q = \sum b_i *t_i^q \in L^q(K^*, \mathfrak{A}),$$

and define

$$(3.4) \qquad (x^p \times *x^q) = \sum_{ij} a_i b_j (t_i^p \times *t_j^q) = \sum a_i b_i .$$

3.41. *If* $x^p \in L^p(K, \mathfrak{A})$, $y^p \in L^p(K, \mathfrak{A})$, *then*

$$(3.41) \qquad (x^p \times D^q y^p) = (x^p \cdot y^p).$$

This follows from the fact that if

$$x^p = \sum a_i t_i^p, \qquad y^p = \sum b_i t_i^p,$$

then

$$D^q y^p = \sum b_i *t_i^q,$$

and so

$$(x^p \times D^q y^p) = \sum a_i b_i = (x^p \cdot y^p).$$

The following two distributive laws are immediate consequences of (3.4):

$$(3.42) \qquad ((x_1^p + x_2^p) \times *x^q) = (x_1^p \times *x^q) + (x_2^p \times *x^q),$$

$$(x^p \times (*x_1^q + *x_2^q)) = (x^p \times *x_1^q) + (x^p \times *x_2^q).$$

Another consequence is

$$(3.43) \qquad (ax^p \times *x^q) = (x^p \times a*x^q) = a(x^p \times *x^q).$$

A fundamental formula in the theory of intersections is

$$(3.44) \qquad (x^p \times \Delta *x^{q+1}) = (-1)^p (\Delta x^p \times *x^{q+1}),$$

or

$$(\Delta x^p \times *x^{q+1}) = (-1)^p (x^p \times \Delta *x^{q+1}).$$

Because of (3.42) and (3.43), we need merely prove (3.44) for the special case $x^p = t_i^p$, $*x^q = *t_j^q$. Hence we need to show only

$$(3.440) \qquad (t_i^p \times \Delta^* t_j^{q+1}) = (-1)^p(\Delta t_i^p \times *t_j^{q+1}).$$

But, setting $\varepsilon_{ik} = (t_i^p : t_k^{p-1})$, we get

$$\Delta t_i^p = \textstyle\sum_k \varepsilon_{ik} t_k^{p-1},$$

$$\Delta^* t_j^{q+1} = (-1)^p \textstyle\sum_h \varepsilon_{hj} {}^*t_h^q.$$

[The second equality follows from (3.31).] Therefore [remembering that $(t_i^p \times *t_i^q) = 1$],

$$(t_i^p \times \Delta^* t_j^{q+1}) = (-1)^p \textstyle\sum_h \varepsilon_{hj}(t_i^p \times *t_h^q)$$

$$= (-1)^p \varepsilon_{ij}(t_i^p \times *t_i^q) = (-1)^p \varepsilon_{ij},$$

$$(\Delta t_i^p \times *t_j^{q+1}) = \textstyle\sum_k \varepsilon_{ik}(t_k^{p-1} \times *t_j^{q+1}) = \varepsilon_{ij}(t_j^{p-1} \times *t_j^{q+1}) = \varepsilon_{ij}.$$

Consequently,

$$(t_i^p \times \Delta^* t_j^{q+1}) = (-1)^p(\Delta t_i^p \times *t_j^{q+1}),$$

which is what we were to prove.

COROLLARY TO (3.44). Let $*x^q$ be a cycle: $\Delta^* x^q = 0$. Then on replacing $p$ by $p + 1$ and hence $q + 1$ by $q$ in (3.44), the left side of this identity becomes zero, and

$$(\Delta x^{p+1} \times *x^q) = 0.$$

But an arbitrary cycle $z^p$ homologous to zero may be written as $\Delta x^{p+1}$. Therefore,

3.451. *If* $*z^q \in Z_\Delta^q(K^*)$, $z^p \in H_\Delta^p(K)$, *then*

$$(z^p \times *z^q) = 0.$$

Analogously,

3.452. *If* $z^p \in Z_\Delta^p(K)$ *and* $*z^q \in H_\Delta^q(K^*)$, *then* $(z^p \times *z^q) = 0$.

For, then $\Delta z^p = 0$, $*z^q = \Delta^* x^{q+1}$ and

$$(z^p \times *z^q) = (z^p \times \Delta^* x^{q+1}) = (-1)^p(\Delta z^p \times *x^{q+1}) = 0.$$

We rewrite 3.451 and 3.452 as one proposition:

3.45. *If at least one of the two cycles* $z^p$ *and* $*z^q$ *is homologous to zero, then*

$$(z^p \times *z^q) = 0.$$

An immediate consequence of 3.45 is:

If $z_1^p \sim z_2^p$ (in $K$), $*z_1^q \sim *z_2^q$ (in $K^*$), then

$$(z_1^p \times *z_1^q) = (z_2^p \times *z_2^q).$$

This makes it possible to define the *intersection number of two homology classes* $\mathfrak{z}^p \in \Delta^p(K)$ *and* $*\mathfrak{z}^q \in \Delta^q(K^*)$ as

$$(\mathfrak{z}^p \times *\mathfrak{z}^q) = (z^p \times *z^q),$$

where $z^p \in \mathfrak{z}^p$, $*z^q \in *\mathfrak{z}^q$ are arbitrary.

REMARK. If the coefficient domain is $J$, Theorem 3.45 may be strengthened. Suppose $*z^q \in Z^q(K^*, J)$, $z^p \in \hat{H}^p(K, J)$. Then there is an integer $c$ such that $cz^p \in H^p(K, J)$, and hence $(cz^p \times *z^q) = 0$. But $(cz^p \times *z^q) = c(z^p \times *z^q)$, so that $(z^p \times *z^q) = 0$. Therefore,

3.46. *If at least one of the cycles* $z^p$ *and* $*z^q$ *is weakly homologous to zero* (in $K$ or $K^*$, respectively), *then* $(z^p \times *z^q) = 0$.

**§3.5. Veblen's theorem.** *For every* $(J, \mathfrak{R})$*-basis* (see VIII, Def. 2.35) $z_1^p, \cdots, z_\pi^p$ *of a complex* $K$ *there is a* $(J, \mathfrak{R})$*-basis* $*z_1^q, \cdots, *z_\pi^q$ *of the complex* $K^*$ *such that*

$$(z_i^p \times *z_j^q) = \delta_{ij} \qquad (i, j = 1, 2, \cdots, \pi),$$

*with* $\pi = \pi^p = \pi^q$ *equal to the pth and qth Betti numbers of the manifold* $\| K \|$.

*Proof.* The proof is based on (3.41). Indeed, by IX, Theorem 3.33, there is a canonical basis

$$x_1^p, \cdots, x_{\tau(p-1)}^p, \qquad y_1^p, \cdots, y_{\sigma(p-1)}^p, \qquad z_1^p, \cdots, z_\pi^p,$$
$$u_1^p, \cdots, u_{\tau(p)}^p, \qquad v_1^p, \cdots, v_{\sigma(p)}^p$$

of the module $L^p(K)$ which includes the cycles $z_1^p, \cdots, z_\pi^p$; we consider the cocycles

$$\bar{z}_1^p, \cdots, \bar{z}_\pi^p \tag{3.52}$$

in the dual basis

$$\bar{u}_1^p, \cdots, \bar{u}_{\tau(p-1)}^p, \qquad \bar{v}_1^p, \cdots, \bar{v}_{\sigma(p-1)}^p, \qquad \bar{z}_1^p, \cdots, \bar{z}_\pi^p, \tag{3.51}$$
$$\bar{x}_1^p, \cdots, \bar{x}_{\tau(p)}^p, \qquad \bar{y}_1^p, \cdots, \bar{y}_{\sigma(p)}^p.$$

Since the isomorphism $D^q$ maps $H_\nabla^p(K)$ onto $H_\Delta^q(K^*)$, no nontrivial linear combination of the cycles

$$D^q\bar{z}_1^p, \cdots, D^q\bar{z}_\pi^p \tag{3.53}$$

is homologous to zero in $K^*$.

In order to show that these cycles form a $(J, \mathfrak{R})$-basis of $K^*$ it is enough to show that each cycle $*z^q \in Z_\Delta^q(K^*)$ differs from a linear combination of the cycles $D^q\bar{z}_h^p$ by an element of $\hat{H}^q(K^*)$.

We shall prove this. First, $*z^q = D^qz^p$, where $z^p \in Z_\nabla^p(K)$. Because of the fundamental properties (see IX, 3.5) of the basis (3.51),

$$z^p = \sum a_i\bar{u}_i^p + \sum b_j\bar{v}_j^p + \sum c_h\bar{z}_h^p;$$

hence (since $D^q$ is an isomorphism)

$$*z^q = D^q z^p = \sum a_i D^q \bar{u}_i^p + \sum b_j D^q \bar{v}_j^p + \sum c_h D^q \bar{z}_h^p.$$

Since some integral multiple of the cocycle

$$\sum a_i \bar{u}_i^p + \sum b_j \bar{v}_j^p$$

is cohomologous to zero in $K$, the same multiple of the cycle

$$*v^q = \sum a_i D^q \bar{u}_i^p + \sum b_j D^q \bar{v}_j^p$$

is homologous to zero in $K^*$, that is, $*z^q - \sum c_h D^q \bar{z}_h^p = *v^q \in \hat{H}^q(K^*)$.

Finally, by (3.41), we have $(z_i^p \times D^q \bar{z}_j^p) = (z_i^p \cdot \bar{z}_j^p) = \delta_{ij}$. This completes the proof of the theorem.

COROLLARY TO VEBLEN'S THEOREM. *Suppose that* $z^p \in Z^p(K)$, *but* $z^p \notin \hat{H}^p(K)$; *then there is a cycle* $*z^q \in Z^q(K^*)$ *such that* $(z^p \times *z^q) \neq 0$.

In fact, if $z_1^p, \cdots, z_\pi^p$ is a $(J, \mathfrak{R})$-basis of $K$, there exist numbers $c_i$, not all zero, such that

$$z^p - \sum c_i z_i^p \in \hat{H}^p(K).$$

Suppose, e.g., that $c_1 \neq 0$. We construct, in accordance with Veblen's theorem, cycles $*z_i^q \in Z_\Delta^q(K^*), i = 1, \cdots, \pi$, such that $(z_i^p \times *z_j^q) = \delta_{ij}$. Then by 3.46,

$$(z^p \dot{\times} *z_1^q) = \sum c_i (z_i^p \times *z_1^q) = c_1 \neq 0.$$

EXERCISE. Prove the analogues of Veblen's theorem and its corollary for the coefficient domains $J_m$, $m$ a prime.

## §4. The combinatorial form of the Alexander duality

**§4.1. Formulation of the theorem.** In this section $K$ will denote a combinatorial $n$-dimensional orientable $h$-manifold, $K_0$ a closed subcomplex of $K$, $K^*$ the complex of the barycentric stars of $K$ and $K_0^*$ the subcomplex of $K^*$ consisting of all the stars dual to the simplexes of $K_0$.

We know (IV, 5.4) that $K_0^*$ is an open, and that therefore $K^* \setminus K_0^*$ is a closed, subcomplex of $K^*$.

*The complex* $K^* \setminus K_0^*$ *is* 0-*torsion free.*

This is true because the complex $Q^*$ consisting of all the 1- and 0-elements of the complex $K^* \setminus K_0^*$ is isomorphic to an unrestricted simplicial 1-complex and has the same 0*th* Betti group as the complex $K^* \setminus K_0^*$.

Finally we denote by $p$ and $q$, as before, two nonnegative integers whose sum is $n$. The purpose of this section is to prove the following theorem, the combinatorial form of the Alexander duality:

THEOREM 4.10. *Let* $\mathfrak{A}$ *be an arbitrary coefficient domain. If* $\nabla^p(K, \mathfrak{A}) = \nabla^{p+1}(K, \mathfrak{A}) = 0$ *for some* $p$, $1 \leq p \leq n - 2$, *then*

$$\nabla^p(K_0, \mathfrak{A}) \approx \Delta^{q-1}(K^* \setminus K_0^*, \mathfrak{A}).$$

*If* $\nabla^{n-1}(K, \mathfrak{A}) = 0$, *then* $\nabla^{n-1}(K_0, \mathfrak{A}) \approx \Delta^{00}(K^* \setminus K_0^*)$ [*if* $\mathfrak{A} = J$, *the number of components of* $K^* \setminus K_0^*$ *is* $\pi^{n-1}(K_0) + 1$]. *Finally, if* $\nabla^1(K, \mathfrak{A}) = 0$, *then* $\Delta^{n-1}(K^* \setminus K_0^*, \mathfrak{A})$ *is the direct sum of* $\pi^0(K_0) - 1$ *groups isomorphic to* $\mathfrak{A}$ [*if* $\mathfrak{A} = J$, *the number of components of* $K_0$ *is equal to*

$$\pi^{n-1}(K^* \setminus K_0^*) + 1$$

*and* $\Theta^{n-1}(K^* \setminus K_0^*) = 0$].

Because of IX, Theorem 3.51, the following more compact statement follows from 4.10:

4.1. THE COMBINATORIAL CASE OF THE ALEXANDER DUALITY IN ITS ORIGINAL FORM. *If* $\Delta_0{}^r(K) = 0$, $0 < r < n$, *then* $\Theta^{p-1}(K_0) \approx \Theta^{q-1}(K^* \setminus K_0^*)$ *and* $\pi^p(K_0) = \pi^{q-1}(K^* \setminus K_0^*)$, $1 \leq p \leq n - 2$. *Furthermore,* $\Theta^{n-1}(K^* \setminus K_0^*) = 0$ *and* $\pi^{n-1}(K^* \setminus K_0^*) = \pi^0(K_0) - 1$; $\Theta^{n-2}(K_0) = 0$ (see the Remark below) *and* $\pi^0(K^* \setminus K_0^*) = \pi^{n-1}(K_0) + 1$.

REMARK. To see that $\Theta^{n-2}(K_0) = 0$, we note that by 4.10, $\nabla_0{}^{n-1}(K_0) = \Theta^{n-2}(K_0) + \Delta_{00}{}^{n-1}(K_0) \approx \Delta^{00}(K^* \setminus K_0^*)$. But $\Delta^{00}(K^* \setminus K_0^*)$ is a free Abelian group; hence $\Theta^{n-2}(K_0) = 0$. The rank of $\Delta_{00}{}^{n-1}(K_0)$ is $\pi^{n-1}(K_0)$ and the rank of $\Delta^{00}(K^* \setminus K_0^*)$ is 1 less than the number of components of $K^* \setminus K_0^*$ (see VIII, 1.62).

Before passing to the proof of Theorem 4.10, we shall deduce several of its consequences and special cases.

If $p = 1$, then $q - 1 = n - 2$, and 4.1 implies that $\Theta^{n-2}(K^* \setminus K_0^*) \approx \Theta^0(K_0)$, i.e., it is the null group. Furthermore, according to IX, Theorem 3.53, $\Theta^{n-1}(K_0)$ is also the null group. Comparing these results with 4.1 and recalling that every $n$-complex is $n$-torsion free, we obtain

4.11. *If* $\Delta_0{}^r(K) = 0$, $0 < r < n$, *then* $K_0$ *and* $K^* \setminus K_0^*$ *are r-torsion free for* $r \geq n - 2$.

A special case of 4.11 is

4.12. *If* $K$ *is a triangulation of a* 3*-sphere and* $K_0$ *is a closed subcomplex of* $K$, *then* $K_0$ *is torsion free in all dimensions.*

Putting $\mathfrak{A} = J_m$ in 4.10, $m$ an arbitrary integer $\geq 2$, and noting (see IX, 4.4) that $\Delta_m{}^r \approx \nabla_m{}^r$, we get

4.13. THE ALEXANDER DUALITY (mod $m$) (COMBINATORIAL CASE). *If* $\Delta_0{}^r(K) = 0$, $0 < r < n$, $\Delta_m{}^p(K_0) \approx \Delta_m{}^{q-1}(K^* \setminus K_0^*)$, $1 \leq p \leq n - 2$; *the groups* $\Delta_m{}^{n-1}(K_0)$ *and* $\Delta_m{}^{00}(K^* \setminus K_0^*)$ *are also isomorphic; finally,* $\Delta_m{}^{n-1}(K^* \setminus K_0^*)$ *is isomorphic to the direct sum of* $\pi^0(K_0) - 1$ *groups* $J_m$.

The following important theorem, to which we give special emphasis, is a special case of 4.1 and 4.13:

4.14. THE COMBINATORIAL FORM OF THE $n$-DIMENSIONAL JORDAN THEOREM. *If* $\Delta_0{}^r(K) = 0$, $0 < r < n$, *then the number of components of* $K^* \setminus K_0^*$ *is equal to* $\pi^{n-1}(K_0) + 1 = \pi_m{}^n(K_0) + 1$, *m an arbitrary prime.*

EXAMPLE FOR THEOREM 4.1. Let $K$ be a triangulation of a 3-sphere and let $K_0$ be a two-dimensional closed combinatorial pseudomanifold which is a subcomplex of $K$ (e.g., a triangulation of a closed surface). Then

1. $\Theta^1(K_0) = 0$ (that is, the pseudomanifold $K_0$ is orientable and torsion free).
2. The 1*st* Betti numbers of $K_0$ and $K^* \setminus K_0^*$ are equal.
3. $K^* \setminus K_0^*$ has two components.

**§4.2. Generalization of Theorem 4.10.** We say that $z_0^p$ is an *extensible* (over $K$) cocycle of $K_0$ if there is a cocycle $z^p$ of $K$ such that $K_0 z^p = z_0^p$. The extensible $p$-cocycles of $K_0$ form a subgroup of $Z_\nabla{}^p(K_0)$; this subgroup will be denoted by $C_K{}^p(K_0)$.

EXAMPLES OF EXTENSIBLE AND NONEXTENSIBLE COCYCLES.

Let $K$ be the triangulation of a torus shown in Fig. 142 (the sides $AA'$ and $A''A'''$, and also $AA''$ and $A'A'''$, are to be identified).

We denote by $K_0$ the complex consisting of the three segments $|t_1{}^1|$, $|t_2{}^1|$, $|t_3{}^1|$ and their vertices $A = A' = A'' = A'''$, $C = C'$, $D = D'$. $K_0$ is therefore a triangulation of the meridian circle $AA' = A''A'''$ of the torus. Let $t_1{}^1$, $t_2{}^1$, $t_3{}^1$ be the orientations of the corresponding segments indicated by arrows in Fig. 142. Since $K_0$ is a 1-complex, every 1-chain of $K_0$ is a cocycle. We shall show that every 1-cocycle of $K_0$ is extensible over $K$. It is enough to prove this assertion for cocycles of the form $t_i{}^1$, $i = 1$, 2, 3, e.g., for $t_1{}^1$. But this cocycle is extensible because the chain $z^1 = t_1{}^1 + t_4{}^1 + t_7{}^1 + t_{19}{}^1 + t_{22}{}^1 + t_{25}{}^1$ (Fig. 142) is a cocycle of $K$ such that $K_0 z^1 = t_1{}^1$.

We now let $K_0$ be the complex consisting of the six segments $|t_1{}^1|$, $|t_2{}^1|$, $|t_3{}^1|$, $|t_4{}^1|$, $|t_5{}^1|$, $|t_6{}^1|$ and their vertices $A = A' = A'' = A'''$, $C = C'$, $D = D'$, $M = M'$, $P$, $Q$. We show that the cocycle $t_1{}^1 - t_4{}^1$ of $K_0$ is nonextensible over $K$. To this end let $z^1$ be any cocycle of $K$ and let

$$\xi_1,\ \xi_2,\ \xi_3;\qquad \eta_1,\ \eta_2,\ \eta_3;\qquad \eta_1',\ \eta_2',\ \eta_3';\qquad \zeta_1,\ \zeta_2,\ \zeta_3$$

be the values of $z^1$ on

$$t_{10}{}^1,\ t_{11}{}^1,\ t_{12}{}^1;\qquad t_1{}^1,\ t_2{}^1,\ t_3{}^1;\qquad t_4{}^1,\ t_5{}^1,\ t_6{}^1;\qquad t_{19}{}^1,\ t_{20}{}^1,\ t_{21}{}^1,$$

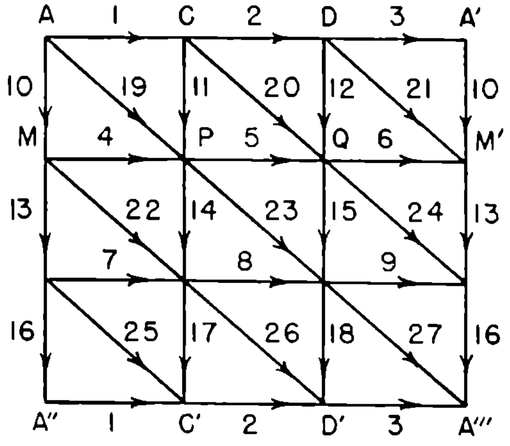


FIG. 142

respectively (Figs. 142, 143). Since the values of the chain $\nabla z^1$ on the six triangles

$$MAP,\ PAC,\ PCQ,\ QCD,\ QDM',\ M'DA'$$

oriented clockwise are zero, we obtain six equations

$$\begin{aligned}
-\xi_1 - \eta_1' + \zeta_1 &= 0,\\
+\xi_2 + \eta_1 - \zeta_1 &= 0,\\
-\xi_2 - \eta_2' + \zeta_2 &= 0,\\
+\xi_3 + \eta_2 - \zeta_2 &= 0,\\
-\xi_3 - \eta_3' + \zeta_3 &= 0,\\
+\xi_1 + \eta_3 - \zeta_3 &= 0.
\end{aligned}$$

Termwise addition yields

$$(II) \qquad \eta_1 + \eta_2 + \eta_3 = \eta_1' + \eta_2' + \eta_3';$$

every cocycle $z^1 \in Z_\nabla{}^1(K)$ satisfies this equation. But if $K_0 z^1 = t_1{}^1 - t_4{}^1$, then $\eta_1 = 1$, $\eta_1' = -1$, $\eta_2 = \eta_3 = \eta_2' = \eta_3' = 0$, which contradicts $(II)$. This proves that the cocycle $t_1{}^1 - t_4{}^1 \in Z_\nabla{}^1(K_0, J)$ is nonextensible.

We consider a third example. Let $K_0$ be the complex consisting of the three sides $|\, t_{11}{}^1 \,|$, $|\, t_{19}{}^1 \,|$, $|\, t_1{}^1 \,|$, and the three vertices of the triangle $PAC$ (Fig. 142). We leave to the reader to prove that the cocycles $t_1{}^1$, $t_{11}{}^1$, $t_{19}{}^1$ of $K_0$ are nonextensible over $K$.

Having given these examples, which we hope have clarified the concept of extensible cocycle, we prove that

$$(4.211) \qquad H_\nabla{}^p(K_0) \subseteq C_K{}^p(K_0)$$

for arbitrary $K$ and $K_0$.

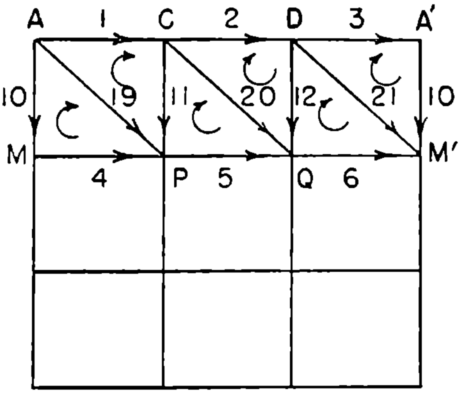


FIG. 143

If

$$z_0^p \in H_\nabla^p(K_0), \qquad z_0^p = \nabla x_0^{p-1}, \qquad x_0^{p-1} \in L^{p-1}(K_0),$$

then

$$\nabla E x_0^{p-1} = z^p \in Z_\nabla^p(K)$$

in the notation of VII, 5.6. Hence, by IX, (1.51),

$$K_0 z^p = K_0 \nabla E x_0^{p-1} = \nabla K_0 E x_0^{p-1} = \nabla x_0^{p-1} = z_0^p,$$

which completes the proof.

Furthermore,

4.21. *If* $\nabla^p(K) = 0$, *then*

$$C_K^p(K_0) \subseteq H_\nabla^p(K_0),$$

*and consequently,*

(4.21) $$C_K^p(K_0) = H_\nabla^p(K_0).$$

To prove (4.21) it is enough to see that if $\nabla^p(K) = 0$,

$$K_0 z^p \in H_\nabla^p(K_0)$$

for every $z^p \in Z_\nabla^p(K)$.

Indeed, by hypothesis, there is a chain $x^{p-1} \in L^{p-1}(K)$ whose coboundary is $z^p$:

$$\nabla x^{p-1} = z^p;$$

hence

$$\nabla K_0 x^{p-1} = K_0 \nabla x^{p-1} = K_0 z^p,$$

and this completes the proof.

We note, finally, that if $\nabla^{p+1}(K) = 0$, the Poincaré duality implies that $\Delta^{q-1}(K) = 0$; hence $\Delta^{q-1}(K^*) = 0$.

For arbitrary $K$ and $K_0 \subset K$ we now denote by $^*Z^r$ the intersection of $Z_\Delta^r(K^* \setminus K_0^*)$ and $H_\Delta^r(K^*)$, that is, *the subgroup of* $Z_\Delta^r(K^* \setminus K_0^*)$ *consisting of all cycles homologous to zero in* $K^*$; then

$$^*Z^0 = Z_\Delta^{(0)}(K^* \setminus K_0^*) \cap H_\Delta^0(K^*).$$

If $\Delta^r(K^*) = 0$ $(r > 0)$ and $\Delta^{(0)}(K^*) = 0$, then $Z_\Delta^r(K^* \setminus K_0^*)$ is clearly identical with $^*Z^r$:

(4.22) $$^*Z^r = Z_\Delta^r(K^* \setminus K_0^*) \qquad (r \geq 0).$$

Moreover, $H_\Delta^r(K^* \setminus K_0^*)$ is always contained in $^*Z^r$.

Hence, if $\nabla^p(K) = \nabla^{p+1}(K) = 0$, then $C_K^p(K_0) = H_\nabla^p(K_0)$ and $^*Z^{q-1} =$

$Z_\Delta^{q-1}(K^* \setminus K_0^*)$; furthermore, under these conditions, $Z_\nabla^p(K_0)/C_K^p(K_0) = Z_\nabla^p(K_0)/H_\nabla^p(K_0) = \nabla^p(K_0)$, and

$$*Z^{q-1}/H_\Delta^{q-1}(K^* \setminus K_0^*)$$
$$= Z_\Delta^{q-1}(K^* \setminus K_0^*)/H_\Delta^{q-1}(K^* \setminus K_0^*) = \Delta^{q-1}(K^* \setminus K_0^*).$$

Therefore, the first part of Theorem 4.10 (in which $p > 0$) is a special case of

THEOREM 4.2. *If $K$ is an $n$-dimensional orientable $h$-manifold, then* $Z_\nabla^p(K_0)/C_K^p(K_0) \approx *Z^{q-1}/H_\Delta^{q-1}(K^* \setminus K_0^*)$, $0 \le p \le n - 1$.

**§4.3. The cases $p = 0$ and $p = n - 1$; prefatory remarks to the proof of Theorem 4.2.** Before proving Theorem 4.2, we shall derive from it the assertions of Theorem 4.10 (and consequently of Theorem 4.1 as well) concerning the groups $\Delta^{n-1}(K^* \setminus K_0^*)$ and $\Delta^{00}(K^* \setminus K_0^*)$.

If $p = n - 1$ and $\nabla^{n-1}(K, \mathfrak{A}) = 0$, then $C_K^{n-1}(K_0) = H_\nabla^{n-1}(K_0)$, so that $Z_\nabla^{n-1}(K_0)/C_K^{n-1}(K_0) = \nabla^{n-1}(K_0)$. Since $K^*$ is connected, $*Z^0 = Z^{00}(K^* \setminus K_0^*)$. By Theorem 4.2,

$$Z_\nabla^{n-1}(K_0)/C_K^{n-1}(K_0) \approx *Z^0/H_\Delta^0(K^* \setminus K_0^*)$$
$$= Z^{00}(K^* \setminus K_0^*)/H_\Delta^0(K^* \setminus K_0^*) = \Delta^{00}(K^* \setminus K_0^*).$$

Hence, $\nabla^{n-1}(K_0) \approx \Delta^{00}(K^* \setminus K_0^*)$.

Passing to the case $p = 0$, we recall that the 0-cocycles of a connected complex $K$ are precisely the constant 0-chains of $K$, that is, those 0-chains which assume a constant value on all the vertices of $K$ (IX, 1.6). Therefore, $Z_\nabla^0(K_0)$ consists of those 0-chains of $K_0$ which are constant on each component of $K_0$. It follows that $Z_\nabla^0(K_0)$ is the direct sum of groups, each isomorphic to $\mathfrak{A}$, and that the number of terms in the direct sum is equal to the number of components of $K_0$, i.e., $\pi^0(K_0)$. The group $C_K^0(K_0)$ consists of those elements of $Z_\nabla^0(K_0)$ which are constant on all of $K_0$. It is then easy to see that $Z_\nabla^0(K_0)/C_K^0(K_0)$ is isomorphic to the direct sum of $\pi^0(K_0) - 1$ groups $\mathfrak{A}$; but, by Theorem 4.2,

$$Z_\nabla^0(K_0)/C_K^0(K_0) \approx *Z^{n-1}/H_\Delta^{n-1}(K^* \setminus K_0^*).$$

Hence the latter is the direct sum of $\pi^0(K_0) - 1$ terms, each isomorphic to $\mathfrak{A}$.

If, as in Theorem 4.10, $\nabla^1(K) = 0$, then $\Delta^{n-1}(K) = 0$, and

$$*Z^{n-1}/H_\Delta^{n-1}(K^* \setminus K_0^*) = \Delta^{n-1}(K^* \setminus K_0^*).$$

Consequently, $\Delta^{n-1}(K^* \setminus K_0^*)$ is the direct sum of $\pi^0(K_0) - 1$ groups isomorphic to $\mathfrak{A}$. This completes the proof.

EXAMPLE FOR THEOREM 4.2. Let $K$ be the triangulation of the torus shown in Fig. 142, and let $\mathfrak{A} = J$. Let us consider three cases:

1°. $K_0$ consists of the three segments $|t_1^1|$, $|t_2^1|$, $|t_3^1|$ and their vertices.

2°. $K_0$ consists of the six segments $|t_1^1|$, $|t_2^1|$, $|t_3^1|$, $|t_4^1|$, $|t_5^1|$, $|t_6^1|$ and their vertices.

3°. $K_0$ consists of the three segments $|t_1^1|$, $|t_{11}^1|$, $|t_{19}^1|$ and their vertices.

In all three cases, the connectedness of $K$ (and hence of $K^*$) implies that $Z_\Delta^0(K^* \setminus K_0^*) = {}^*Z^0$, ${}^*Z^0/H_\Delta^0(K^* \setminus K_0^*) = \Delta^{00}(K^* \setminus K_0^*)$.

In the first case, $Z_\nabla^1(K_0) = C_K^1(K_0)$, so that $Z_\nabla^1(K_0)/C_K^1(K_0) = 0$. Hence by Theorem 4.2, $\Delta^{00}(K^* \setminus K_0^*) = 0$, i.e., $K^* \setminus K_0^*$ is connected.

In the second and third cases, $C_K^1(K_0)$ is a proper subgroup of $Z_\nabla^1(K_0)$, so that $Z_\nabla^1(K_0)/C_K^1(K_0) \neq 0$, and according to Theorem 4.2 the complex $K^* \setminus K_0^*$ is not connected. It is easy to see that in both cases the group $Z_\nabla^1(K_0)/C_K^1(K_0)$ is infinite cyclic, and hence the complex $K^* \setminus K_0^*$ consists of two components.

It is left to the reader to investigate various examples of the application of Theorem 4.2 when the complex $K$ is a triangulation of a three-dimensional torus or of the topological product of a 2-sphere and a 1-sphere.

The basic technique used in the proof of Theorem 4.2 given below is the isomorphic mapping $D^q$ of $L^p(K)$ onto $L^q(K^*)$ defined in 3.3. As before, we denote the isomorphism of $L^q(K^*)$ onto $L^p(K)$ inverse to $D^q$ by $D_p$ and note the following obvious property of the isomorphisms $D^q$ and $D_p$:

4.31. *If $Q$ is a subcomplex of $K$ and $Q^*$ is the subcomplex of $K^*$ consisting of all the barycentric stars dual to elements of $Q$, then*

$$Q^*D^qx^p = D^qQx^p,$$

$$QD_py^q = D_pQ^*y^q$$

*for all $x^p \in L^p(K)$ and $y^q \in L^q(K^*)$.*

In the sequel elements of $K$, $K_0$, $K^*$, $K_0^*$ will be denoted by $T$, $T_0$, ${}^*T$, ${}^*T_0$, respectively, with supplementary indices added whenever necessary; $T_i^p$ and ${}^*T_i^q$ or $T_{i(0)}^p$ and ${}^*T_{i(0)}^q$ will always stand for pairs of dual elements; the orientations of $T$ and ${}^*T$ will be written as $t$ and ${}^*t$.

**§4.4. The isomorphism $\Delta D^q$ of $Z_\nabla^p(K_0)/C_K^p(K_0)$ onto the group ${}^*Z^{q-1}/H_\Delta^{q-1}(K^* \setminus K_0^*)$.** We assign the chain $\Delta D^q z_0^p \in H_\Delta^{q-1}(K^*)$ to each chain $z_0^p \in Z_\nabla^p(K_0)$. In virtue of the fundamental property of $D^q$ [see (3.32) and (3.320)],

$$\pm(\Delta D^q z_0^p \cdot {}^*t_{i(0)}^{q-1}) = (D^{q-1}\nabla z_0^p \cdot {}^*t_{i(0)}^{q-1})$$

$$= (\nabla z_0^p \cdot t_{i(0)}^{p+1}) = 0$$

for arbitrary $|{}^*t_{i(0)}^{q-1}| \subset K_0^*$; that is,

$$\Delta D^q z_0^p \in Z_\Delta^{q-1}(K^* \setminus K_0^*). \tag{4.41}$$

Moreover, since $\Delta D^q z_0^p \in H_\Delta^{q-1}(K^*)$,

$$\Delta D^q z_0^p \in {}^*Z^{q-1}.$$

4.41. *If the cycle* $\Delta D^q z_0^p$ *is assigned to each cocycle* $z_0^p$ *of* $K_0$, *the result is a homomorphism* $\Delta D^q$ *of* $Z_\nabla^p(K_0)$ *into* $*Z^{q-1}$.

We must prove:

4.42. *The homomorphism* $\Delta D^q$ *of* $Z_\nabla^p(K_0)$ *into* $*Z^{q-1}$ *induces a homomorphism (denoted by the same symbol) of* $Z_\nabla^p(K_0)/C_K^p(K_0)$ *into the group* $*Z^{q-1}/H_\Delta^{q-1}(K^* \setminus K_0^*)$.

4.43. *The induced homomorphism of Theorem* 4.42 *is an isomorphism.*

4.44. *The induced isomorphism of Theorem* 4.43 *is onto.*

To prove 4.42 it is enough to show

4.420. *If* $z_0^p \in C_K^p(K_0)$, *then* $\Delta D^q z_0^p \in H_\Delta^{q-1}(K^* \setminus K_0^*)$.

For the proof of 4.43 it sufficies to show

4.430. *If* $z_0^p \in Z_\nabla^p(K_0)$ *and* $\Delta D^q z_0^p \in H_\Delta^{q-1}(K^* \setminus K_0^*)$, *then*

$$z_0^p \in C_K^p(K_0).$$

To prove 4.44 it is enough to prove

4.440. *For every* $u^{q-1} \in *Z^{q-1}$ *there exists a cocycle* $z_0^p \in Z_\nabla^p(K_0)$ *such that* $\Delta D^q z_0^p \sim u^{q-1}$ *in* $K^* \setminus K_0^*$.

To clarify 4.420 and 4.430 see Fig. 144, where $K_0$ is the segment $(e_0e_1)$ and its two vertices. Furthermore, $z_0^1 = |e_0e_1|$ is a cocycle of $K_0$ extensible over $K$ (the chain whose value is 1 on the segments marked with arrows

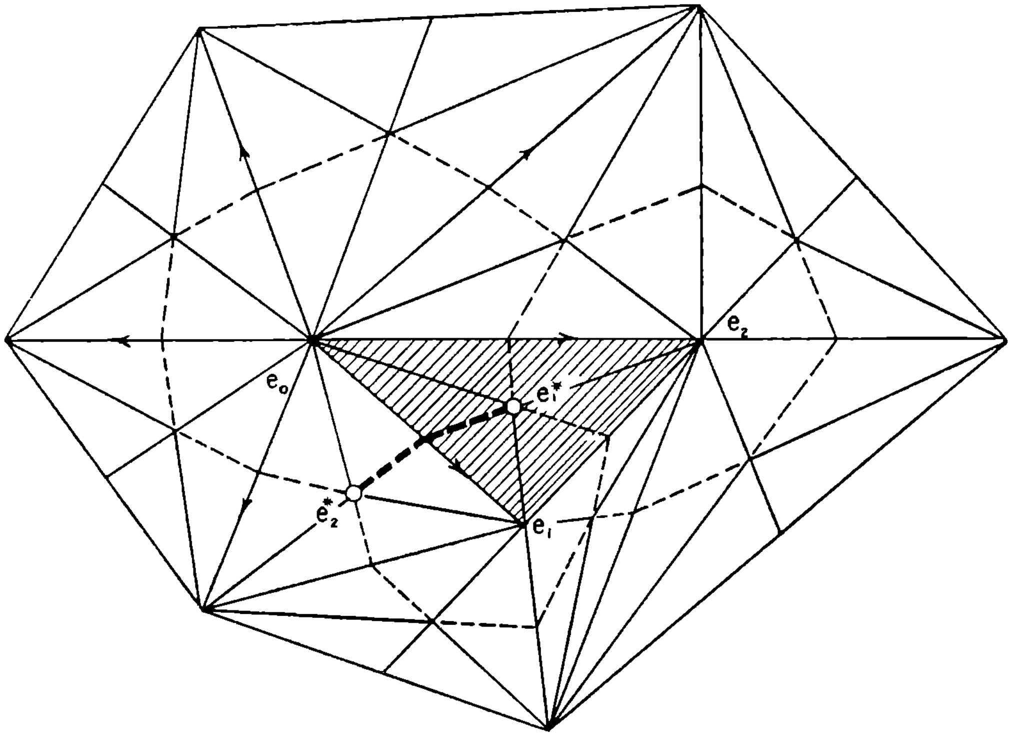


FIG. 144

and 0 on the remaining segments in Fig. 144 is a cocycle of $K$ which coincides with $z_0^1$ on $K_0$).

REMARK. If $K_0$ is the complex consisting of all the sides and vertices of the triangle $(e_0e_1e_2)$, the cocycle

$$z_0^p = |e_0e_1| \in Z_\nabla^p(K_0), \qquad p = 1,$$

is not extensible over $K$, in agreement with the fact that $\Delta D^q z_0^p$ is not homologous to zero in $K^* \setminus K_0^*$ [the points $e_1^*$ and $e_2^*$ are separated by the boundary of the triangle $(e_0e_1e_2)$ and hence by the complex $K_0^*$].

*Proof of* 4.420 (Fig. 144). Suppose $z_0^p \in C_K^p(K_0)$, so that $z_0^p = K_0 z^p$, where $z^p \in Z_\nabla^p(K)$. Then

$$D^q z^p \in Z_\Delta^p(K^*),$$

$$D^q z^p = x^q + y^q,$$

where

$$x^q = K_0^* D^q z^p, \qquad y^q = (K^* \setminus K_0^*) D^q z^p$$

and

$$0 = \Delta D^q z^p = \Delta x^q + \Delta y^q,$$

that is,

$$\Delta(-y^q) = \Delta x^q = \Delta K_0^* D^q z^p.$$

But, in virtue of 4.31,

$$K_0^* D^q z^p = D^q K_0 z^p,$$

so that

$$\Delta(-y^q) = \Delta D^q K_0 z^p = \Delta D^q z_0^p.$$

But $-y^q \in L^q(K^* \setminus K_0^*)$, so that

$$\Delta D^q z_0^p \in H_\Delta^{q-1}(K^* \setminus K_0^*),$$

which completes the proof.

*Proof of* 4.430 (Fig. 144). Suppose $z_0 \in Z_\nabla^p(K_0)$ and

$$\Delta D^q z_0^p \in H_\Delta^{q-1}(K^* \setminus K_0^*).$$

We must find a $z^p \in Z_\nabla^p(K)$ such that $z_0^p = K_0 z^p$.

To this end we first choose a $y^q \in L^q(K^* \setminus K_0^*)$ such that

$$\Delta y^q = \Delta D^q z_0^p;$$

hence

$$D^q z_0^p - y^q \in Z_\Delta^q(K^*).$$

We set

$$z^p = D_p(D^q z_0^p - y^q) \in Z_\nabla^p(K),$$

and prove that $z^p$ is the required cocycle, i.e., that

$$z_0^p = K_0 z^p.$$

We have first

$$z^p = D_p(D^q z_0^p - y^q) = D_p D^q z_0^p - D_p y^q = z_0^p - D_p y^q.$$

Furthermore, since $y^q \in L^q(K^* \setminus K_0^*)$,

$$D_p y^q \in L^p(K \setminus K_0),$$

so that $K_0 D_p y^q = 0$ and $K_0 z^p = K_0(z_0^p - D_p y^q) = K_0 z_0^p = z_0^p$, q.e.d.

*Proof of* 4.440. Suppose (Fig. 145)

$$u^{q-1} \in {}^*Z^{q-1}, \qquad \Delta x^q = u^{q-1}, \qquad x^q \in L^q(K^*).$$

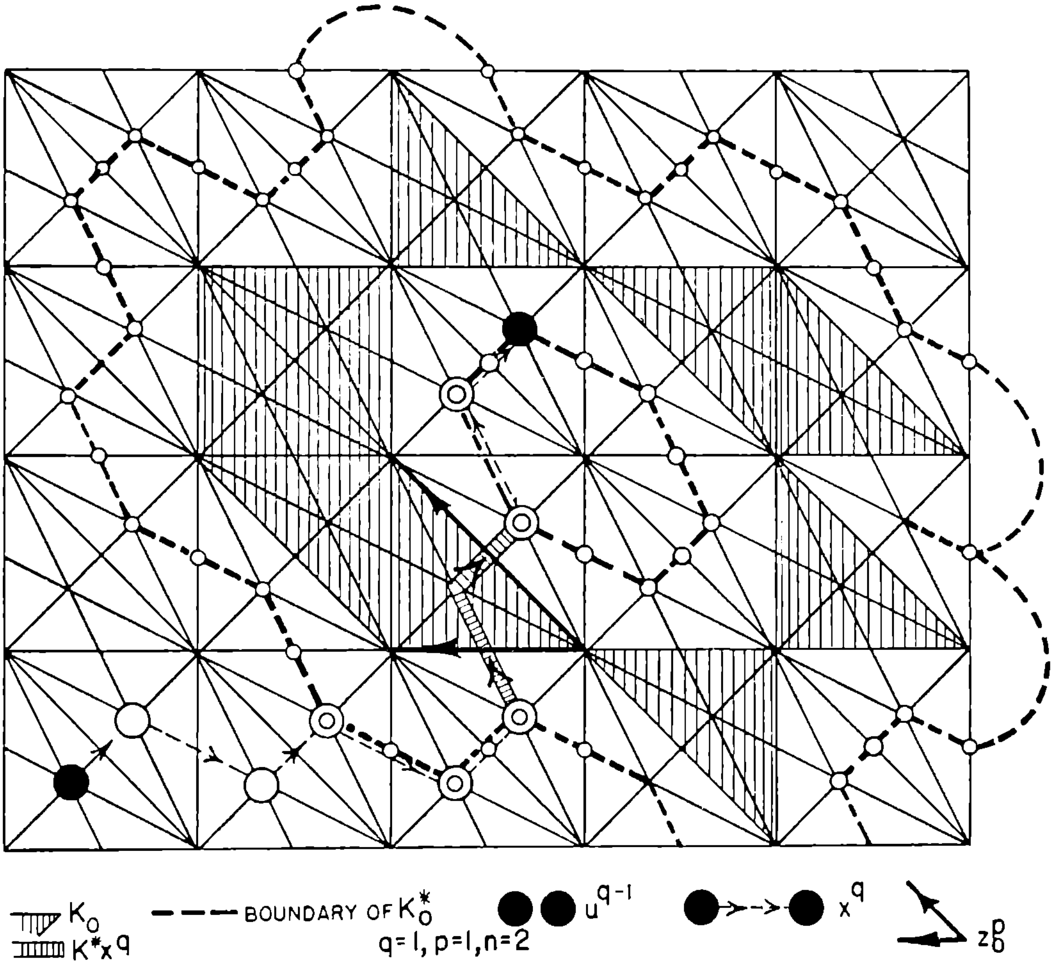


Fig. 145

Then

$$x^q = K_0{}^*x^q + (K^* \setminus K_0{}^*)x^q,$$

and

$$u^{q-1} = \Delta x^q = \Delta K_0{}^*x^q + \Delta(K^* \setminus K_0{}^*)x^q;$$

or (since $K_0{}^*x^q = K_0{}^*D^qD_px^q = D^qK_0D_px^q$)

$$u^{q-1} - \Delta D^qK_0D_px^q = \Delta(K^* \setminus K_0{}^*)x^q;$$

or, finally,

$$u^{q-1} \sim \Delta D^q z_0{}^p \quad \text{in} \quad K^* \setminus K_0{}^*,$$

where

$$z_0{}^p = K_0D_px^q \in L^p(K_0).$$

To show that $z_0{}^p \in Z_\nabla{}^p(K_0)$, we calculate $\nabla z_0{}^p$ on $K_0$:

$$\nabla z_0{}^p = \nabla K_0D_px^q = K_0\nabla D_px^q = \pm K_0D_{p+1}\Delta x^q = \pm K_0D_{p+1}u^{q-1} = 0.$$

This proves 4.440 and completes the proof of Theorem 4.2.

§**4.5. Definition and simplest properties of the linking number** $\mathfrak{v}(u^p, u^{q-1})$ **of two cycles** $u^p \in H_\Delta{}^p(K)$, $u^{q-1} \in H_\Delta{}^{q-1}(K^*)$. All the discussion of this subsection is based on the fundamental formulas of 3.4:

$$(3.44) \qquad \begin{aligned} (x^p \times \Delta y^{q+1}) &= (-1)^p(\Delta x^p \times y^{q+1}), \\ (\Delta x^{p+1} \times y^q) &= (-1)^{p+1}(x^{p+1} \times \Delta y^q) \end{aligned}$$

for arbitrary

$$\begin{aligned} x^p &\in L^p(K), & x^{p+1} &\in L^{p+1}(K), \\ y^q &\in L^q(K^*), & y^{q+1} &\in L^{q+1}(K^*), \end{aligned}$$

and on Theorem 3.45, which is a consequence of these formulas:

3.45. *If at least one of the cycles* $z^p \in Z_\Delta{}^p(K)$, $z^q \in Z_\Delta{}^q(K^*)$ *is homologous to zero, then* $(z^p \times z^q) = 0$.

An immediate consequence is

4.51. *If two chains* $x_1{}^{p+1} \in L^{p+1}(K)$, $x_2{}^{p+1} \in L^{p+1}(K)$ *have a common boundary* $\Delta x_1{}^{p+1} = \Delta x_2{}^{p+1} = u^p$, *then*

$$(x_1{}^{p+1} \times u^{q-1}) = (x_2{}^{p+1} \times u^{q-1})$$

*for arbitrary* $u^{q-1} \in H_\Delta{}^{q-1}(K^*)$.

For,

$$x_1{}^{p+1} - x_2{}^{p+1} \in Z_\Delta{}^{p+1}(K);$$

hence

$$((x_1^{p+1} - x_2^{p+1}) \times u^{q-1}) = 0,$$

that is,

$$(x_1^{p+1} \times u^{q-1}) = (x_2^{p+1} \times u^{q-1}).$$

Theorem 4.51 enables us to state the following

DEFINITION 4.5. *Suppose* $u^p \in H_\Delta{}^p(K)$ *and* $u^{q-1} \in H_\Delta{}^{q-1}(K^*)$. *The linking number of the cycles* $u^p$ *and* $u^{q-1}$ *is the number*

$$\mathfrak{v}(u^p, u^{q-1}) = (x^{p+1} \times u^{q-1}),$$

*where* $x^{p+1} \in L^{p+1}(K)$ *is an arbitrary chain such that*

$$\Delta x^{p+1} = u^p.$$

Theorem 4.51 implies that $\mathfrak{v}(u^p, u^{q-1})$ is independent of the choice of the chain $x^{p+1}$ satisfying the condition $\Delta x^{p+1} = u^p$.

We prove several simple properties of the linking numbers:

4.52. *If* $u^p \in H_\Delta{}^p(K)$, $u^{q-1} \in H_\Delta{}^{q-1}(K^*)$,

$$u^{q-1} = \Delta y^q, \qquad y^q \in L^q(K^*),$$

*then*

$$\mathfrak{v}(u^p, u^{q-1}) = (-1)^{p+1}(u^p \times y^q). \tag{4.52}$$

Indeed, suppose that $x^{p+1} \in L^{p+1}(K)$ and $\Delta x^{p+1} = u^p$. Formula (3.44) implies that

$$\begin{aligned}\mathfrak{v}(u^p, u^{q-1}) &= (x^{p+1} \times \Delta y^q) = (-1)^{p+1}(\Delta x^{p+1} \times y^q) \\ &= (-1)^{p+1}(u^p \times y^q),\end{aligned}$$

and the proof is complete.

The definition of $\mathfrak{v}(u^p, u^{q-1})$ implies further:

4.531. Let $K_0$ be a closed subcomplex of $K$, and let $K_0^* \subseteq K^*$ be the complex consisting of the barycentric stars dual to the elements of $K_0$; if $u^p \in H_\Delta{}^p(K_0)$, then $\mathfrak{v}(u^p, u^{q-1}) = 0$ for an arbitrary cycle

$$u^{q-1} \in Z_\Delta{}^{q-1}(K^* \setminus K_0^*)$$

homologous to zero in $K^*$.

For, by hypothesis,

$$u^p = \Delta x_0{}^{p+1},$$

where

$$x_0{}^{p+1} \in L^{p+1}(K_0).$$

Hence,

$$\mathfrak{v}(u^p, u^{q-1}) = (x_0^{p+1} \times u^{q-1}) = 0.$$

The following proposition may be proved in a similar way (by means of 4.52):

4.532. If $u^{q-1} \in H_\Delta^{q-1}(K^* \setminus K_0^*)$, then $\mathfrak{v}(u^p, u^{q-1}) = 0$ for any cycle $u^p \in Z_\Delta^p(K_0)$ homologous to zero in $K$.

Theorems 4.531 and 4.532 immediately imply

4.53. *Let* $u_1^p$ *and* $u_2^p$ *be two cycles of* $K_0$, *homologous to zero in* $K$ *and homologous to each other in* $K_0$; *if* $u_1^{q-1}$ *and* $u_2^{q-1}$ *are two cycles of* $K^* \setminus K_0^*$, *homologous to zero in* $K^*$ *and homologous to each other in* $K^* \setminus K_0^*$, *then*

$$\mathfrak{v}(u_1^p, u_1^{q-1}) = \mathfrak{v}(u_2^p, u_2^{q-1})$$

(see Fig. 146, where $u_1^p$ is the outer boundary of the left hand triangle, oriented by the indicated arrows, $u_2^p$ is the boundary of the little triangle, oriented as indicated, and $u_1^{q-1} = u_2^{q-1}$).

Now let $Z_{\Delta K}^p(K_0)$ be the subgroup of $Z_\Delta^p(K_0)$ consisting of all the cycles homologous to zero in $K$. Then for arbitrary $\mathfrak{u}^p \in Z_{\Delta K}^p(K_0)/H_\Delta^p(K_0)$, $\mathfrak{u}^{q-1} \in {}^*Z^{q-1}/H_\Delta^{q-1}(K^* \setminus K_0^*)$ we set

$$\mathfrak{v}(\mathfrak{u}^p, \mathfrak{u}^{q-1}) = \mathfrak{v}(u^p, u^{q-1}), \tag{4.54}$$

where $u^p$ and $u^{q-1}$ are arbitrary elements of the cosets $\mathfrak{u}^p$, $\mathfrak{u}^{q-1}$, respectively; according to 4.53, the number $\mathfrak{v}(\mathfrak{u}^p, \mathfrak{u}^{q-1})$ is independent of the choice of the cycles $u^p \in \mathfrak{u}^p$ and $u^{q-1} \in \mathfrak{u}^{q-1}$.

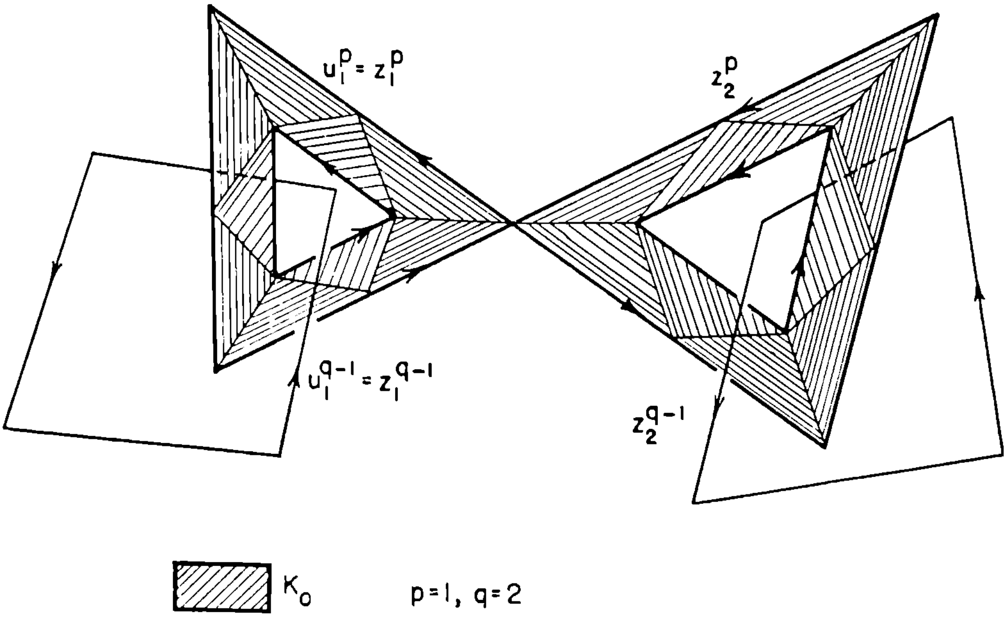


FIG. 146

In particular, if $\Delta^p(K) = 0$ and $\nabla^{p+1}(K) = 0$, then $\Delta^{q-1}(K^*) = 0$, and it follows that

$$Z_{\Delta K}{}^p(K_0)/H_\Delta{}^p(K_0) = \Delta^p(K_0),$$

$$*Z^{q-1}/H_\Delta{}^{q-1}(K^* \setminus K_0^*) = \Delta^{q-1}(K^* \setminus K_0^*);$$

then (4.54) gives the *linking number* $\mathfrak{v}(u^p, u^{q-1})$ for arbitrary

$$u^p \in \Delta^p(K_0), u^{q-1} \in \Delta^{q-1}(K^* \setminus K_0^*).$$

**§4.6. Theorem on linked systems of cycles.** In this subsection $\mathfrak{A}$ will stand for either the field of rational numbers or one of the fields $J_m$, $m$ a prime.

We shall say that two systems of cycles

$$z_1{}^p, \cdots, z_s{}^p, \qquad z_i{}^p \in Z^p(K_0, \mathfrak{A})$$

and

$$z_1{}^{q-1}, \cdots, z_s{}^{q-1}, \qquad z_i{}^{q-1} \in Z^{q-1}(K^* \setminus K_0^*, \mathfrak{A})$$

are *linked* if $\mathfrak{v}(z_i{}^p, z_j{}^{q-1}) = \delta_{ij}$ (Fig. 146).

THEOREM 4.61 (Pontryagin). *For every system of lirh* (see VII, 6.420) *cycles*

$$z_1{}^p, \cdots, z_s{}^p, \qquad z_1{}^p \in Z^p(K_0, \mathfrak{A}) \tag{4.611}$$

*of $K_0$ it is possible to construct a system of cycles*

$$z_1{}^{q-1}, \cdots, z_s{}^{q-1}, \qquad z_i{}^{q-1} \in Z^{q-1}(K^* \setminus K_0^*, \mathfrak{A}) \tag{4.612}$$

*of $K^* \setminus K_0^*$ which is linked with the system* (4.611).

*Proof.* We extend the cycles (4.611) to a canonical basis

$$x_i{}^p, y_j{}^p, z_1{}^p, \cdots, z_s{}^p, u_k{}^p, v_l{}^p \tag{4.613}$$

of the module $L^p(K_0, \mathfrak{A})$. In the basis

$$\bar{u}_i{}^p, \bar{v}_j{}^p, \bar{z}_1{}^p, \cdots, \bar{z}_s{}^p, \bar{x}_k{}^p, \bar{y}_l{}^p$$

dual to (4.613) we consider the cocycles $\bar{z}_1{}^p, \cdots, \bar{z}_s{}^p$, and construct the cycles

$$\Delta D^q \bar{z}_1{}^p, \cdots, \Delta D^q \bar{z}_s{}^p$$

of $K^* \setminus K_0^*$. By (3.41),

$$(z_i{}^p \times D^q \bar{z}_j{}^p) = (z_i{}^p \cdot \bar{z}_j{}^p) = \delta_{ij};$$

hence, by (4.52)

$$\mathfrak{v}(z_i{}^p, \Delta D^q \bar{z}_j{}^p) = (-1)^{p+1}(z_i{}^p \times D^q \bar{z}_j{}^p) = (-1)^{p+1}\delta_{ij}.$$

The cycles

$$z_i^{q-1} = (-1)^{p+1}\Delta D^q \bar{z}_i^p$$

are the required cycles.

4.611. For every system of lirh *integral* cycles

$$(4.611)_J \qquad z_1^p, \cdots, z_s^p$$

of $K_0$ it is possible to construct a system of *integral* cycles

$$(4.612)_J \qquad z_1^{q-1}, \cdots, z_s^{q-1}$$

of $K^* \setminus K_0^*$ such that

$$\mathfrak{v}(z_i^p, z_i^{q-1}) \neq 0, \qquad \mathfrak{v}(z_i^p, z_j^{q-1}) = 0 \qquad (i \neq j).$$

*Proof.* Consider the cycles $(4.611)_J$ as cycles with rational coefficients and construct, in accordance with Theorem 4.61, a system of rational cycles $z_1'^{q-1}, \cdots, z_s'^{q-1}$ linked with the system $(4.611)_J$. Then, denoting by $c_i$ the common denominator of the fractions which appear as the coefficients of the cycles $z_j'^{q-1}$ and putting $z_i^{q-1} = c_i z_i'^{q-1}$, we obtain the required system $(4.612)_J$.

THEOREM 4.62 (Pontryagin). *Suppose the integral cycles*

$$(4.621) \qquad z_1^p, \cdots, z_\pi^p$$

*form a* $(J, \mathfrak{R})$*-basis of* $K_0$. *Then it is possible to construct a system of integral cycles*

$$(4.622) \qquad z_1^{q-1}, \cdots, z_\pi^{q-1}$$

*of* $K^* \setminus K_0^*$ *which form a* $(J, \mathfrak{R})$*-basis of* $K^* \setminus K_0^*$ *and are linked with the system* (4.621).

*Proof.* We note first that since the system (4.621) is a $(J, \mathfrak{R})$-basis of $K_0$, it can be extended to a canonical basis

$$(4.613) \qquad x_i^p, y_j^p, z_h^p, u_k^p, v_l^p$$

of $L^p(K_0, \mathfrak{A})$ (see IX, Theorem 3.33).

Then, as in the proof of Theorem 4.61, we consider in the dual basis

$$(4.623) \qquad \bar{u}_i^p, \bar{v}_j^p, \bar{z}_h^p, \bar{x}_k^p, \bar{y}_l^p$$

the cocycles

$$\bar{z}_1^p, \cdots, \bar{z}_\pi^p$$

and construct the cycles

$$(4.624) \qquad \Delta D^q \bar{z}_1^p, \cdots, \Delta D^q \bar{z}_\pi^p.$$

In exactly the same way as in the proof of Theorem 4.61 we see that the cycles (4.624), each multiplied by $(-1)^{p+1}$, form a system linked with the system (4.621).

The system (4.624) consists of $\pi = \pi^{q-1}(K^* \setminus K_0^*)$ cycles; therefore, to show that these cycles form a $(J, \mathfrak{N})$-basis of $K^* \setminus K_0^*$ it is enough to prove that each cycle $z^{q-1} \in Z^{q-1}(K^* \setminus K_0^*)$ differs from a linear combination of the cycles $\Delta D^q \bar{z}_h^p$ by an element of $\hat{H}^{q-1}(K^* \setminus K_0^*)$.

We shall prove this. First, by (4.440),

$$z^{q-1} = \Delta D^q z_0^p + u^{q-1}, \tag{4.625}$$

where

$$z_0^p \in Z_\nabla^p(K_0), \qquad u^{q-1} \in H_\Delta^{q-1}(K^* \setminus K_0^*) \subseteq \hat{H}^{q-1}(K^* \setminus K_0^*).$$

By the fundamental properties of the basis (4.623),

$$z_0^p = \sum a_i \bar{u}_i^p + \sum b_j \bar{v}_j^p + \sum c_h \bar{z}_h^p;$$

hence, since $\Delta D^q$ is a homomorphism of $Z_\nabla^p(K_0)$ into $Z_\Delta^{q-1}(K^* \setminus K_0^*)$,

$$\Delta D^q z_0^p = \sum a_i \Delta D^q \bar{u}_i^p + \sum b_j \Delta D^q \bar{v}_j^p + \sum c_h \Delta D^q \bar{z}_h^p.$$

Since some integral multiple of the cocycle

$$v_0^p = \sum a_i \bar{u}_i^p + \sum b_j \bar{v}_j^p$$

is homologous to zero in $K_0$ and is consequently contained in $C_K^p(K_0)$, it follows by 4.420 that the same integral multiple of the cycle

$$v^{q-1} = \Delta D^q v_0^p = \sum a_i \Delta D^q \bar{u}_i^p + \sum b_j \Delta D^q \bar{v}_j^p$$

is homologous to zero in $K^* \setminus K_0^*$, i.e.,

$$v^{q-1} \in \hat{H}^{q-1}(K^* \setminus K_0^*);$$

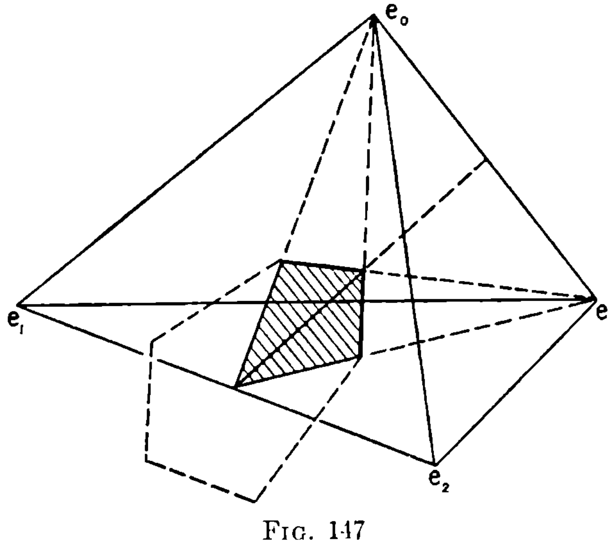


FIG. 147

hence $z^{q-1} - \sum c_h \Delta D^q \bar{z}_h^{\,p} = u^{q-1} + v^{q-1} \in \hat{H}^{q-1}(K^* \setminus K_0^*)$. This completes the proof.

In conclusion, we give an example illustrating the basic construction used in the proof of Theorems 4.61 and 4.62. The construction consisted in forming the dual basis of a canonical basis of $K_0 \subset K$ and operating on the former with the operator $\Delta D^q$.

EXAMPLE. Let $K$ be a triangulation of the 3-sphere $S^3$; we think of the 3-sphere as the usual three-dimensional space compactified by a single point at infinity. Let the complex $K_0 \subset K$ consist of the three sides and vertices of the triangle $(e_1e_2e_3)$ (Fig. 147).

We take a canonical basis of the module $L_0^1(K_0)$ (see IX, 3.4, Example 1):

$$y_1^1 = t_1^1 = |\, e_2e_3 \,|, \qquad y_2^1 = t_2^1 = |\, e_3e_1 \,|, \qquad z_1^1 = t_1^1 + t_2^1 + t_3^1,$$

where $t_3^1 = |\, e_1e_2 \,|$. In the dual basis (IX, 3.5, Examples of $\nabla$-bases)

$$\bar{v}_1^1 = t_1^1 - t_3^1, \qquad \bar{v}_2^1 = t_2^1 - t_3^1, \qquad \bar{z}_1^1 = t_3^1$$

we take the cocycle $\bar{z}_1^1 = t_3^1 = |\, e_1e_2 \,|$ and construct the cycle $\Delta D^q \bar{z}_1^1$ (here $q = 2, p = 1, n = 3$). The cycle $\Delta D^q \bar{z}_1^1$ is shown in Fig. 147 as an oriented polygon (the boundary of the oriented barycentric star $*t^2 = D^q t_3^1$).

*Chapter XIV*

# THE $\nabla$-GROUPS (COHOMOLOGY GROUPS) OF COMPACTA AND THE ALEXANDER-PONTRYAGIN DUALITY

## §1. $\nabla$-groups (cohomology groups) of bicompacta

### §1.1. Spectra and their $\nabla$-groups.

DEFINITION 1.1. An infinite partially ordered set $\Theta$ is said to be *directed* by $>$ if $x_1, x_2 \in \Theta$ implies that there exists an $x_3 \in \Theta$ such that $x_3 > x_1$ and $x_3 > x_2$.

DEFINITION 1.12. Let $\mathfrak{K}$ be a directed partially ordered set of finite unrestricted simplicial complexes $K_\alpha$. If $K_\beta > K_\alpha$ in $\mathfrak{K}$, we write simply $\beta > \alpha$.

Suppose further that one or more simplicial mappings $S_\alpha{}^\beta$ of $K_\beta$ into $K_\alpha$ is associated with every pair of complexes $K_\alpha$, $K_\beta$ of $\mathfrak{K}$ such that $K_\beta > K_\alpha$. We call these mappings *projections* and require that they satisfy the following conditions:

1°. If $S_\alpha{}^\beta$ and $'S_\alpha{}^\beta$ are two projections of $K_\beta$ into $K_\alpha$, then for every simplex $T_\beta \subset K_\beta$ there is a simplex $T_\alpha \in K_\alpha$ such that both $S_\alpha{}^\beta T_\beta$ and $'S_\alpha{}^\beta T_\beta$ are faces of $T_\alpha$.

2°. If $\gamma > \beta > \alpha$ and if $S_\alpha{}^\beta$, $S_\beta{}^\gamma$ are projections of $K_\beta$ into $K_\alpha$ and $K_\gamma$ into $K_\beta$, respectively, then the simplicial mapping $S_\alpha{}^\beta S_\beta{}^\gamma$ of $K_\gamma$ into $K_\alpha$ is a projection.

If both of these conditions are satisfied, we call the pair consisting of the partially ordered set $\mathfrak{K}$ and the simplicial mappings $S_\alpha{}^\beta$ a *projective spectrum*, or simply a *spectrum*.

Theorem 9.4 of VII and Theorem 6.3 of IX imply

1.13. If $S_\alpha{}^\beta$ and $'S_\alpha{}^\beta$ are two projections of $K_\beta$ into $K_\alpha$, $K_\alpha$, $K_\beta \in \mathfrak{K}$, then

$$\bar{S}_\beta{}^\alpha z_\alpha \backsim {}'\bar{S}_\beta{}^\alpha z_\alpha \quad \text{in } K_\beta$$

for arbitrary $z_\alpha \in Z_\nabla{}^r(K_\alpha)$.

Hence,

1.14. Suppose that $z_\alpha \in Z_\nabla{}^r(K_\alpha)$, $z_\beta \in Z_\nabla{}^r(K_\beta)$ and that $\gamma > \alpha, \gamma > \beta$. If there exist two projections $S_\alpha{}^\gamma$, $S_\beta{}^\gamma$ such that

$$\bar{S}_\gamma{}^\alpha z_\alpha \backsim \bar{S}_\gamma{}^\beta z_\beta \quad \text{in } K_\gamma, \tag{1.14}$$

then (1.14) remains valid for every other pair of projections $'S_\alpha{}^\gamma$, $'S_\beta{}^\gamma$.

DEFINITION 1.15. Two cocycles $z_\alpha \in Z_\nabla{}^r(K_\alpha)$, $z_\beta \in Z_\nabla{}^r(K_\beta)$ are said to be *cohomologous in* $\mathfrak{K}$:

$$z_\alpha \backsim z_\beta \quad \text{in } \mathfrak{K},$$

if there is a $\gamma$, $\gamma > \alpha$, $\gamma > \beta$, such that

$$\bar{S}_{\gamma}{}^{\alpha} z_{\alpha} \backsim \bar{S}_{\gamma}{}^{\beta} z_{\beta} \quad \text{in } K_{\gamma},$$

i.e., if

$$\bar{S}_{\gamma}{}^{\alpha} z_{\alpha} - \bar{S}_{\gamma}{}^{\beta} z_{\beta} \in H_{\nabla}{}^{r}(K_{\gamma}).$$

It is easily seen that the relation of cohomology between cocycles of the complexes of a spectrum $\mathfrak{K}$ is reflexive, symmetric and transitive. We give a short proof of transitivity, the only property which perhaps requires it.

Suppose

$$z_{\alpha} \in Z_{\nabla}{}^{r}(K_{\alpha}), \qquad z_{\beta} \in Z_{\nabla}{}^{r}(K_{\beta}), \qquad z_{\gamma} \in Z_{\nabla}{}^{r}(K_{\gamma})$$

and

$$\bar{S}_{\lambda}{}^{\alpha} z_{\alpha} \backsim \bar{S}_{\lambda}{}^{\beta} z_{\beta} \quad \text{in } K_{\lambda}, \qquad S_{\mu}{}^{\beta} z_{\beta} \backsim S_{\mu}{}^{\gamma} z_{\gamma} \quad \text{in } K_{\mu},$$

$$\lambda > \alpha, \qquad \lambda > \beta, \qquad \mu > \beta, \qquad \mu > \gamma.$$

Choose $K_{\nu}$ so that $\nu > \lambda$, $\nu > \mu$. Then

$$\bar{S}_{\nu}{}^{\lambda} \bar{S}_{\lambda}{}^{\alpha} z_{\alpha} \backsim \bar{S}_{\nu}{}^{\lambda} \bar{S}_{\lambda}{}^{\beta} z_{\beta}, \qquad \bar{S}_{\nu}{}^{\mu} \bar{S}_{\mu}{}^{\beta} z_{\beta} \backsim \bar{S}_{\nu}{}^{\mu} \bar{S}_{\mu}{}^{\gamma} z_{\gamma} \quad \text{in } K_{\nu}.$$

Since $\bar{S}_{\nu}{}^{\lambda} \bar{S}_{\lambda}{}^{\beta} z_{\beta} \backsim \bar{S}_{\nu}{}^{\mu} \bar{S}_{\mu}{}^{\beta} z_{\beta}$ in $K_{\nu}$ by (1.13), it follows that

$$\cdot \ \bar{S}_{\nu}{}^{\lambda} \bar{S}_{\lambda}{}^{\alpha} z_{\alpha} \backsim \bar{S}_{\nu}{}^{\mu} \bar{S}_{\mu}{}^{\gamma} z_{\gamma} \quad \text{in } K_{\nu},$$

i.e.,

$$z_{\alpha} \backsim z_{\gamma} \quad \text{in } \mathfrak{K}.$$

*Therefore, the cohomology relation partitions the set of all r-cocycles of all the complexes $K_{\alpha}$ of a spectrum $\mathfrak{K}$ into r-dimensional cohomology classes $\zeta^{r}$ of $\mathfrak{K}$* (further indices will be attached to $\zeta^{r}$ whenever necessary).

We shall now define addition for the cohomology classes. We note first that two arbitrary cohomology classes $\zeta_1{}^{r}$, $\zeta_2{}^{r}$ contain elements $z_{\alpha}{}'$ and $z_{\alpha}{}''$ belonging to the same complex $K_{\alpha} \in \mathfrak{K}$:

$$z_{\alpha}{}' \in Z_{\nabla}{}^{r}(K_{\alpha}), \qquad z_{\alpha}{}'' \in Z_{\nabla}{}^{r}(K_{\alpha}).$$

In fact, choose arbitrary

$$z_{\alpha_1} \in \zeta_1{}^{r}, \qquad z_{\alpha_2} \in \zeta_2{}^{r},$$

let $K_{\alpha} \in \mathfrak{K}$ be such that $\alpha > \alpha_1$, $\alpha > \alpha_2$ and set

$$z_{\alpha}{}' = \bar{S}_{\alpha}{}^{\alpha_1} z_{\alpha_1} \in \zeta_1{}^{r}, \qquad z_{\alpha}{}'' = \bar{S}_{\alpha}{}^{\alpha_2} z_{\alpha_2} \in \zeta_2{}^{r}.$$

To define the sum of two cohomology classes $\zeta_1{}^{r}$ and $\zeta_2{}^{r}$ choose arbitrary cocycles $z_{\alpha}{}' \in \zeta_1{}^{r}$ and $z_{\alpha}{}'' \in \zeta_2{}^{r}$ of the same $K_{\alpha}$; the cohomology class $\zeta_3{}^{r}$

containing the cocycle $z_\alpha' + z_\alpha''$ is, by definition, the sum of $\zeta_1^r$ and $\zeta_2^r$. It is easily verified that this sum depends only on $\zeta_1^r$ and $\zeta_2^r$ and not on the choice of $z_\alpha' \in \zeta_1^r$, $z_\alpha'' \in \zeta_2^r$.

To show this, suppose

$$z_\beta' \in \zeta_1^r, \qquad z_\beta'' \in \zeta_2^r, \qquad z_\beta' \in Z_\nabla^r(K_\beta), \qquad z_\beta'' \in Z_\nabla^r(K_\beta).$$

We must show that

$$\bar{S}_\gamma^\alpha(z_\alpha' + z_\alpha'') \backsim \bar{S}_\gamma^\beta(z_\beta' + z_\beta'')$$

in some $K_\gamma$, $\gamma > \alpha$, $\gamma > \beta$. Since $z_\alpha' \backsim z_\beta'$ in $\mathfrak{K}$, there exists a $K_{\gamma_1}$, $\gamma_1 > \alpha$, $\gamma_1 > \beta$ such that

$$\bar{S}_{\gamma_1}^\alpha z_\alpha' \backsim \bar{S}_{\gamma_1}^\beta z_\beta' \quad \text{in } K_{\gamma_1};$$

since $z_\alpha'' \backsim z_\beta''$ in $\mathfrak{K}$, there is a $K_{\gamma_2}$, $\gamma_2 > \alpha$, $\gamma_2 > \beta$, such that

$$\bar{S}_{\gamma_2}^\alpha z_\alpha'' \backsim \bar{S}_{\gamma_2}^\beta z_\beta''.$$

Take $\gamma > \gamma_1$, $\gamma > \gamma_2$. Then

$$\bar{S}_\gamma^{\gamma_1}\bar{S}_{\gamma_1}^\alpha z_\alpha' \backsim \bar{S}_\gamma^{\gamma_1}\bar{S}_{\gamma_1}^\beta z_\beta' \quad \text{in } K_\gamma,$$

$$\bar{S}_\gamma^{\gamma_2}\bar{S}_{\gamma_2}^\alpha z_\alpha'' \backsim \bar{S}_\gamma^{\gamma_2}\bar{S}_{\gamma_2}^\beta z_\beta'' \quad \text{in } K_\gamma,$$

so that

$$\bar{S}_\gamma^{\gamma_1}\bar{S}_{\gamma_1}^\alpha z_\alpha' + \bar{S}_\gamma^{\gamma_2}\bar{S}_{\gamma_2}^\alpha z_\alpha'' \backsim \bar{S}_\gamma^{\gamma_1}\bar{S}_{\gamma_1}^\beta z_\beta' + \bar{S}_\gamma^{\gamma_2}\bar{S}_{\gamma_2}^\beta z_\beta'' \quad \text{in } K_\gamma.$$

Since

$$\bar{S}_\gamma^{\gamma_1}\bar{S}_{\gamma_1}^\alpha z_\alpha' \backsim \bar{S}_\gamma^\alpha z_\alpha', \qquad \bar{S}_\gamma^{\gamma_2}\bar{S}_{\gamma_2}^\alpha z_\alpha'' \backsim \bar{S}_\gamma^\alpha z_\alpha'',$$

$$\bar{S}_\gamma^{\gamma_1}\bar{S}_{\gamma_1}^\beta z_\beta' \backsim \bar{S}_\gamma^\beta z_\beta', \qquad \bar{S}_\gamma^{\gamma_2}\bar{S}_{\gamma_2}^\beta z_\beta'' \backsim \bar{S}_\gamma^\beta z_\beta''$$

(see 1.13), it follows that

$$\bar{S}_\gamma^\alpha z_\alpha' + \bar{S}_\gamma^\alpha z_\alpha'' \backsim \bar{S}_\gamma^\beta z_\beta' + \bar{S}_\gamma^\beta z_\beta'' \quad \text{in } K_\gamma.$$

This completes the proof.

The addition of $r$-dimensional cohomology classes of a spectrum $\mathfrak{K}$ is commutative and, as is easily seen, associative. The zero of this addition is the $r$-dimensional $\nabla$-class $O^r$ containing the identities of all the groups $\nabla^r(K_\alpha)$; it is obvious that

$$\zeta^r + O^r = \zeta^r$$

for an arbitrary $\nabla$-class $\zeta^r$.

We note that the $\nabla$-class $O^r$ consists of all the $r$-cocycles $z_\alpha^r$ of all possible complexes $K_\alpha \in \mathfrak{K}$ for which there is a $K_\beta \in \mathfrak{K}$, $\beta > \alpha$, such that

$$\bar{S}_\beta^\alpha z_\alpha^r \backsim 0 \quad \text{in } K_\beta.$$

Furthermore, for each ∇-class $\zeta^r$ there exists a ∇-class $-\zeta^r$ such that

$$\zeta^r + (-\zeta^r) = O^r;$$

to find $-\zeta^r$ it is sufficient to take any $z_\alpha^r \in \zeta^r$ and define $-\zeta^r$ as the ∇-class containing the cocycle $-z_\alpha^r$ (it is easily seen that $-\zeta^r$ is independent of the choice of $z_\alpha^r \in \zeta^r$ and that $-\zeta^r$ consists of all cocycles of the form $-z_\alpha^r$, where $z_\alpha^r \in \zeta^r$).

Hence,

*The set of ∇-classes of a spectrum $\mathcal{K}$, with addition of ∇-classes defined as above, is a group called the r-dimensional ∇-group or the $\nabla^r$-group (rth cohomology group) of the spectrum $\mathcal{K}$ and is denoted by* $\nabla^r(\mathcal{K})$.

**§1.2. The homomorphism $\bar{S}^\alpha$.** Two cocycles $z_\alpha \in Z_\nabla{}^r(K_\alpha)$ and $z_\alpha' \in Z_\nabla{}^r(K_\alpha)$ cohomologous in $K_\alpha$ are cohomologous in $\mathcal{K}$; therefore, every cohomology class of $K_\alpha \in \mathcal{K}$ is contained in some cohomology class of the spectrum $\mathcal{K}$. As a result, there is a natural homomorphism $\bar{S}^\alpha$ of $\nabla^r(K_\alpha)$ into $\nabla^r(\mathcal{K})$.

We shall require the following very special theorem in the sequel:

1.21. *Let $\mathcal{K}$ be a spectrum with the following property: for every $K_\alpha$, $K_\beta \in \mathcal{K}$, $\beta > \alpha$, and every projection $S_\alpha^\beta$, $\bar{S}_\beta^\alpha$ induces an isomorphism of $\nabla^r(K_\alpha)$ onto $\nabla^r(K_\beta)$. Then $\nabla^r(K_\alpha)$ is isomorphic to $\nabla^r(\mathcal{K})$ for every $K_\alpha \in \mathcal{K}$.*

*Proof.* It is sufficient to show that every cohomology class $\zeta^r$ of the spectrum $\mathcal{K}$ contains precisely one cohomology class of an arbitrary $K_\alpha \in \mathcal{K}$.

Hence, suppose that $\zeta^r \in \nabla^r(\mathcal{K})$ and $K_\alpha \in \mathcal{K}$ are given.

Suppose that $z_{\alpha_1} \in \zeta^r$, $z_{\alpha_1} \in Z_\nabla{}^r(K_{\alpha_1})$; choose a $K_\beta \in \mathcal{K}$ such that $\beta > \alpha$, $\beta > \alpha_1$. Then $z_\beta = \bar{S}_\beta^{\alpha_1} z_{\alpha_1} \in \zeta^r$. Since a homomorphism $\bar{S}_\beta^\alpha$ of $L^r(K_\alpha)$ into $L^r(K_\beta)$ induces an isomorphism of $\nabla^r(K_\alpha)$ onto $\nabla^r(K_\beta)$, there exists a cocycle $z_\alpha \in Z_\nabla{}^r(K_\alpha)$ such that $\bar{S}_\beta^\alpha z_\alpha \backsim z_\beta$ in $K_\beta$. Obviously, $z_\alpha \in \zeta^r$. Hence, $\zeta^r \in \nabla^r(\mathcal{K})$ contains the cohomology class of $z_\alpha \in Z_\nabla{}^r(K_\alpha)$. It remains to be shown that $z_\alpha' \in Z_\nabla{}^r(K_\alpha)$ and $z_\alpha' \backsim z_\alpha$ in $\mathcal{K}$ imply that $z_\alpha' \backsim z_\alpha$ in $K_\alpha$. But $z_\alpha' \backsim z_\alpha$ in $\mathcal{K}$ implies the existence of a $K_\beta \in \mathcal{K}$, $\beta > \alpha$, such that $\bar{S}_\beta^\alpha z_\alpha' \backsim \bar{S}_\beta^\alpha z_\alpha$ in $K_\beta$; since $\bar{S}_\beta^\alpha$ induces an isomorphism of $\nabla^r(K_\alpha)$ onto $\nabla^r(K_\beta)$, it follows that

$$\bar{S}_\beta^\alpha z_\alpha' \backsim \bar{S}_\beta^\alpha z_\alpha \quad \text{in } K_\beta$$

implies

$$z_\alpha' \backsim z_\alpha \quad \text{in } K_\alpha,$$

which was to be proved.

**§1.3. The ∇-groups of bicompacta** (see I, 4.3). Let $\Phi$ be an infinite bicompactum. We consider the directed partially ordered set $\Theta$ of all finite open coverings (see I, 8.1) of $\Phi$; the coverings will be denoted by $\alpha$, $\beta$, $\gamma$, etc.; $K_\alpha$ will denote the nerve of the covering $\alpha$ and, by definition, $K_\beta > K_\alpha$ if, and only if, $\beta > \alpha$.

If $\beta > \alpha$, assign to each element of the covering $\beta$ any element of the covering $\alpha$ which contains it; the result is a mapping of $\beta$ into $\alpha$ called a *projection* of $\beta$ into $\alpha$. Every projection of $\beta$ into $\alpha$ induces a simplicial mapping of the complex $K_\beta$ into the complex $K_\alpha$; the induced simplicial mappings are also called projections of $K_\beta$ into $K_\alpha$. They satisfy Condition 1° of 1.1. To see this, let $S_\alpha^\beta$ and $'S_\alpha^\beta$ be two projections of $K_\beta$ into $K_\alpha$, and let

$$T_\beta = (e_0^\beta \cdots e_r^\beta)$$

be any simplex of $K_\beta$. Denoting a vertex of the nerve and the element of the covering corresponding to it by the same letter, we see that $e_i^\beta$ is contained in both $S_\alpha^\beta e_i^\beta$ and $'S_\alpha^\beta e_i^\beta$; hence

$$S_\alpha^\beta e_0^\beta \cap {'S_\alpha^\beta} e_0^\beta \cap \cdots \cap S_\alpha^\beta e_r^\beta \cap {'S_\alpha^\beta} e_r^\beta \supseteq e_0^\beta \cap \cdots \cap e_r^\beta \neq 0.$$

Then both $S_\alpha^\beta T_\beta$ and $'S_\alpha^\beta T_\beta$ are faces of the simplex $T_\alpha \in K_\alpha$ whose vertices are $S_\alpha^\beta e_0^\beta, {'S_\alpha^\beta} e_0^\beta, \cdots, S_\alpha^\beta e_r^\beta, {'S_\alpha^\beta} e_r^\beta$.

The projections $S_\alpha^\beta$ also obviously satisfy Condition 2° of 1.1.

Hence,

*The pair consisting of the partially ordered set $\mathfrak{K}$ of the nerves $K_\alpha$ of all the finite open coverings $\alpha$ of a bicompactum $\Phi$ and the projections defined above is a projective spectrum called the full projective spectrum of $\Phi$. The cohomology groups ($\nabla$-groups) of the spectrum $\mathfrak{K}$ are called the cohomology groups ($\nabla$-groups) of $\Phi$ and are denoted by $\nabla^r(\Phi)$.*

REMARK 1. All the arguments and definitions of 1.3 can be repeated using closed coverings $\alpha, \beta, \gamma, \cdots$ of $\Phi$ instead of open coverings of $\Phi$; it can be proved that the cohomology groups defined by means of closed coverings are isomorphic to those defined by means of open coverings.

REMARK 2. The definition of the cohomology groups given above is formally applicable to arbitrary topological spaces; the cohomology groups defined by open and closed coverings are also isomorphic in normal spaces. However, our definition can be considered as final only for bicompacta; it is expedient to use another definition even in the case of locally bicompact normal spaces (see Aleksandrov [k; l; f (see vol. 1, Bibliography)]).

**§1.4. The group $\nabla^{00}$.** Let $K_\alpha$ be a finite unrestricted simplicial complex. We denote by $Z_\nabla^{00}(K_\alpha)$ the subgroup of the group $Z_\nabla^0(K)$ consisting of all 0-cocycles constant on $K_\alpha$ (see XIII, 4.3, the case $p = 0$).

REMARK 1. The elements of $Z_\nabla^{00}(K_\alpha)$ are 0-cocycles extensible over an arbitrary unrestricted simplicial complex $K \supset K_\alpha$.

Now let $K_\alpha$ and $K_\beta$ be elements of a spectrum $\mathfrak{K}$, and suppose that $z_\alpha^0 \in Z_\nabla^0(K_\alpha)$, $z_\beta^0 \in Z_\nabla^0(K_\beta)$. We shall say (see Def. 1.15) that $z_\alpha^0$ and $z_\beta^0$ are contained in the same $\nabla^{00}$-class of the spectrum $\mathfrak{K}$ if there is a $\gamma, \gamma > \alpha, \gamma > \beta$, such that

$$\bar{S}_\gamma^\alpha z_\alpha^0 - \bar{S}_\gamma^\beta z_\beta^0 \in Z_\nabla^{00}(K_\gamma).$$

It is easily verified, as in 1.1, that this is a proper definition. With addition for the $\nabla^{00}$-classes defined in exactly the same way as for the ∇-classes, the set of $\nabla^{00}$-classes becomes a group which we call the $\nabla^{00}$-group of the spectrum $\mathcal{K}$ and denote by $\nabla^{00}(\mathcal{K})$. If $\mathcal{K}$ is the full spectrum of a bicompactum $\Phi$, then we call the group $\nabla^{00}(\mathcal{K})$ the group $\nabla^{00}(\Phi)$.

REMARK 2. We leave the proof of the following theorem to the reader:

1.41. If $\mathfrak{A}$ is the coefficient domain and the bicompactum $\Phi$ has a finite number $\pi^0$ of components, then $\nabla^{00}(\Phi)$ is the direct sum of $\pi^0 - 1$ groups, each isomorphic to $\mathfrak{A}$.

The proof is based on XIII, 4.3 and on the fact that every $K_\alpha \in \mathcal{K}$ consists of no more than $\pi^0$ components, while for every $K_\alpha$ there is a $K_\beta > K_\alpha$ with $\pi^0$ components.

## §2. Cofinal suborders of spectra. The case of compacta

### §2.1. Cofinal suborders of spectra.

DEFINITION 2.11. A partially ordered set $\Theta'$ is said to be a *suborder* of a partially ordered set $\Theta$ if every element of $\Theta'$ is an element of $\Theta$ and if

$$x_1 \in \Theta', \qquad x_2 \in \Theta', \qquad x_1 > x_2 \quad \text{in } \Theta'$$

implies that

$$x_1 > x_2 \quad \text{in } \Theta.$$

A suborder $\Theta'$ of a directed partially ordered set $\Theta$ is said to be *cofinal* with $\Theta$ if $\Theta'$ is a directed partially ordered set and if $x \in \Theta$ implies that there exists a $y \in \Theta'$ such that

$$y > x \quad \text{in } \Theta.$$

DEFINITION 2.12. A spectrum $\mathcal{K}'$ is said to be cofinal with a spectrum $\mathcal{K}$ if the following conditions are satisfied:

1°. The spectrum $\mathcal{K}'$, as a partially ordered set consisting of complexes, is cofinal with the partially ordered set $\mathcal{K}$.

2°. If $K_\beta > K_\alpha$ in $\mathcal{K}'$, then every projection of $K_\beta$ into $K_\alpha$ in $\mathcal{K}'$ is a projection in $\mathcal{K}$.

Every spectrum $\mathcal{K}'$ cofinal with the full projective spectrum $\mathcal{K}$ of a bicompactum $\Phi$ is called a projective spectrum of $\Phi$.

THEOREM 2.13. *If a spectrum $\mathcal{K}'$ is cofinal with a spectrum $\mathcal{K}$, then* $\nabla^r(\mathcal{K}) \approx \nabla^r(\mathcal{K}')$.

*Proof.* The set of all $r$-cocycles $z_\alpha^r$ of the complexes of $\mathcal{K}'$ is a subset of the set of $r$-cocycles of the complexes of $\mathcal{K}$; since $z_\alpha^r \backsim z_\beta^r$ in $\mathcal{K}'$ implies that $z_\alpha^r \backsim z_\beta^r$ in $\mathcal{K}$, every ∇-class of $\mathcal{K}'$ is contained in a (unique) ∇-class of $\mathcal{K}$. It is not hard to see that the resulting natural mapping of $\nabla^r(\mathcal{K}')$ into $\nabla^r(\mathcal{K})$ is a homomorphism.

We shall show that this homomorphism is an isomorphism onto. To this end, it is sufficient to prove that every $\nabla$-class of $\mathcal{K}$ contains precisely one $\nabla$-class of $\mathcal{K}'$.

Let $\zeta$ be a $\nabla$-class of $\mathcal{K}$ and suppose $z_\alpha \in \zeta$. Since $\mathcal{K}'$ is cofinal with $\mathcal{K}$, there is a complex $K_\beta \in \mathcal{K}'$ such that $K_\beta > K_\alpha$ in $\mathcal{K}$. Then the cocycle $z_\beta = \bar{S}_\beta{}^\alpha z_\alpha$ of $K_\beta \in \mathcal{K}'$ is contained in a $\nabla$-class $\zeta'$ of $\mathcal{K}'$; since $\zeta'$ has an element $z_\beta$ in common with $\zeta$, it is contained in $\zeta$. Hence every $\nabla$-class of $\mathcal{K}$ contains at least one $\nabla$-class of $\mathcal{K}'$. It remains to be shown that no $\nabla$-class of $\mathcal{K}$ contains more than one $\nabla$-class of $\mathcal{K}'$. To this end, it suffices to prove that two cocycles

$$z_\alpha \in Z_\nabla{}^r(K_\alpha), \qquad z_\beta \in Z_\nabla{}^r(K_\beta), \qquad K_\alpha \in \mathcal{K}', \qquad K_\beta \in \mathcal{K}',$$

cohomologous in $\mathcal{K}$ are also cohomologous in $\mathcal{K}'$. Suppose that

$$z_\alpha \backsim z_\beta \quad \text{in } \mathcal{K}.$$

Then there is a complex $K_{\gamma_1} \in \mathcal{K}$, $\gamma_1 > \alpha$, $\gamma_1 > \beta$, such that

$$\bar{S}_{\gamma_1}{}^\alpha z_\alpha \backsim \bar{S}_{\gamma_1}{}^\beta z_\beta \quad \text{in } K_{\gamma_1}.$$

Choose a complex $K_{\gamma_2}$ in $\mathcal{K}'$ such that $K_{\gamma_2} > K_{\gamma_1}$ in $\mathcal{K}$; such a complex $K_{\gamma_2} \in \mathcal{K}'$ exists, since $\mathcal{K}'$ is cofinal with $\mathcal{K}$.

Now choose a complex $K_\gamma \in \mathcal{K}'$ such that $K_\gamma > K_\alpha$, $K_\beta$, $K_{\gamma_2}$ in $\mathcal{K}'$. The complex $K_\gamma$ exists since $\mathcal{K}'$ is a directed partially ordered set.

Projections $\bar{S}_\gamma{}^\alpha$, $\bar{S}_\gamma{}^\beta$ are defined in $\mathcal{K}'$; since both mappings are at the same time projections in $\mathcal{K}$ and since $\gamma > \alpha$, $\gamma > \beta$, $\gamma > \gamma_1$ in $\mathcal{K}$, it follows that

$$\bar{S}_\gamma{}^\alpha z_\alpha \backsim \bar{S}_\gamma{}^{\gamma_1} \bar{S}_{\gamma_1}{}^\alpha z_\alpha \backsim \bar{S}_\gamma{}^{\gamma_1} \bar{S}_{\gamma_1}{}^\beta z_\beta \backsim \bar{S}_\gamma{}^\beta z_\beta$$

in $K_\gamma$, i.e.,

$$\bar{S}_\gamma{}^\alpha z_\alpha \backsim \bar{S}_\gamma{}^\beta z_\beta \quad \text{in } K_\gamma.$$

This completes the proof.

COROLLARY. *The* $\nabla^r$*-groups of a bicompactum* $\Phi$ *are isomorphic to the* $\nabla^r$*-groups of an arbitrary spectrum of* $\Phi$.

### §2.2. The case of $\Phi$ a compactum.

THEOREM 2.21. *Let* $\Theta$ *be the partially ordered set of all open coverings* (see I, 8.1) *of an infinite compactum* $\Phi$; *and let* $\Theta'$ *be a suborder of* $\Theta$ *satisfying the following condition: for every* $\epsilon > 0$, $\Theta'$ *contains an* $\epsilon$*-covering of* $\Phi$. *Then* $\Theta'$ *is cofinal with* $\Theta$.

REMARK 1. The converse is obvious: if $\Theta'$ is cofinal with $\Theta$, than $\Theta'$ contains an $\epsilon$-covering of $\Phi$ for every $\epsilon > 0$.

*Proof of Theorem* 2.21. Let $\alpha$ be an arbitrary covering of $\Phi$ and choose an $\alpha_1 > \alpha$, $\alpha_1 \in \Theta$. There exists (see I, Lemma 8.34) an $\epsilon > 0$ such that every

$\epsilon$-covering of $\Phi$ is a refinement of $\alpha_1$ and is therefore $> \alpha$. If $\gamma \in \Theta'$ is an arbitrary $\epsilon$-covering, we see that $\gamma > \alpha$.

It remains to be proved that $\Theta'$ is directed. If $\alpha, \beta \in \Theta'$, choose a covering $\alpha_1 \in \Theta$ greater than both of these and determine the corresponding $\epsilon > 0$ and $\gamma$ as above.

Remark 2. As the partially ordered set $\Theta'$ of Theorem 2.21 we can take an arbitrary sequence of coverings

$$\Theta' = \{\alpha_1, \alpha_2, \cdots, \alpha_k, \cdots\}, \tag{2.2}$$

where $\alpha_k$ is an $\epsilon_k$-covering and $\lim \epsilon_k = 0$. We may assume that $\alpha_{k+1}$ is a refinement of $\alpha_k$; hence we can suppose that $\Theta'$ is a countable ordered set of the same order type as the natural numbers. The spectrum $\mathfrak{K}'$ corresponding to the set $\Theta'$ is cofinal with the full projective spectrum $\mathfrak{K}$ of $\Phi$, and is therefore a spectrum of $\Phi$. It is countable and has the same order type as the natural numbers.

Next, if the sequence (2.2) is chosen so that for arbitrary $k$ every element of $\alpha_{k+1}$ is contained in precisely one element of $\alpha_k$ (see Remark 3), we obtain a spectrum $\mathfrak{K}'$ of $\Phi$ which contains exactly one projection $S_{\alpha_h}{}^{\alpha_k}$ of $\alpha_k$ into $\alpha_h$ for each pair of coverings $\alpha_h$ and $\alpha_k$, $k > h$.

Finally, if $\dim \Phi = n$, we can choose $n$-complexes as the elements of the spectrum $\mathfrak{K}'$. It follows, in particular, that

2.22 *If* $\dim \Phi = n$, *and* $r > n$, *then* $\nabla^r(\Phi) = 0$.

Remark 3. To obtain a sequence (2.2) satisfying the condition that for all $k$ every element of $\alpha_{k+1}$ is contained in precisely one element of $\alpha_k$, it is enough to take any sequence of subdivisions (see VI, 4)

$$\alpha_1', \alpha_2', \cdots, \alpha_k', \cdots$$

(with each $\alpha_k'$ a closed covering) and to proceed as follows: replace the elements of the covering $\alpha_1'$ by neighborhoods of these elements so small that these neighborhoods form an open covering $\alpha_1$ similar to the covering $\alpha_1'$ and so that every element of $\alpha_2'$ is contained in a single element of $\alpha_1$. Then replace the elements of $\alpha_2'$ by sufficiently small neighborhoods of these elements, etc.

**§2.3. The case of $\Phi$ a polyhedron.** Let $K$ be a triangulation of a polyhedron $\Phi$. Consider the sequence of complexes

$$K, K_1, K_2, \cdots, K_n, \cdots,$$

each of which is the barycentric subdivision of the preceding. Denote by $\alpha_\nu$ the open covering of $\Phi$ whose elements are the open stars of the triangulation $K_\nu$ (see IV, Def. 3.20). Then $K_\nu$ is the nerve of $\alpha_\nu$, so that we may regard the notation $K_\nu$ as an abbreviation of $K_{\alpha_\nu}$.

Let $e_j^{\nu+1}$ be an arbitrary vertex of the complex $K_{\nu+1}$, and denote by

$S_\nu^{\nu+1} e_j^{\nu+1} = e_i^\nu$ a vertex of the carrier of the point $e_j^{\nu+1}$ in $K_\nu$. The mapping $S_\nu^{\nu+1}$ is a normal displacement of $K_{\nu+1}$ into $K_\nu$ (see X, 3.1), so that the open star $O_j^{\nu+1}$ of $e_j^{\nu+1}$ is contained in the open star of $e_j^\nu$ (Fig. 148). Hence $S_\nu^{\nu+1}$ is a projection of $\alpha_{\nu+1}$ into $\alpha_\nu$.

This defines the projections in the projective spectrum $\mathfrak{K}$ consisting of all the complexes $K_\nu$. Since the diameters of the elements of $\alpha_\nu$ approach zero with increasing $\nu$, the spectrum $\mathfrak{K}$ is, according to Theorem 2.21, a spectrum of $\Phi$. Therefore $\nabla^r(\Phi)$ is isomorphic to $\nabla^r(\mathfrak{K})$. But, by X, Theorem 4.3, the simplicial mappings $S_\nu^{\nu+1}$ induce isomorphic mappings of $\nabla^r(K_\nu)$ into $\nabla^r(K_{\nu+1})$; therefore, by Theorem 1.21, $\nabla^r(\mathfrak{K})$ is isomorphic to every $\nabla^r(K_\nu)$, so that it is isomorphic to $\nabla^r(K)$. Hence we have proved

THEOREM 2.4. *If $\Phi$ is a polyhedron and $K$ is a triangulation of $\Phi$, then $\nabla^r(\Phi)$ is isomorphic to $\nabla^r(K)$.*

This proves the invariance of the groups $\nabla^r(K)$, and hence of the groups $\Delta^r(K)$, where $K$ is a polyhedral complex.

REMARK. If the compactum $\Phi$ is finite, the above definition of the group $\nabla^r(\Phi)$ is not applicable, since the partially ordered set of all open coverings of $\Phi$ is finite. But in that case $\Phi$ is a 0-complex and the definition of the group $\nabla$ given in IX, 1.6 can be used. Hence if $\Phi$ is finite and consists of $\pi$ points, then:

the group $\nabla^0(\Phi, \mathfrak{A})$ is the direct sum of $\pi$ groups isomorphic to $\mathfrak{A}$;

the group $\nabla^{00}(\Phi, \mathfrak{A})$ (the only one of interest) is the direct sum of $\pi - 1$ groups isomorphic to $\mathfrak{A}$;

the group $\nabla^r(\Phi, \mathfrak{A}) = 0$ for $r > 0$.

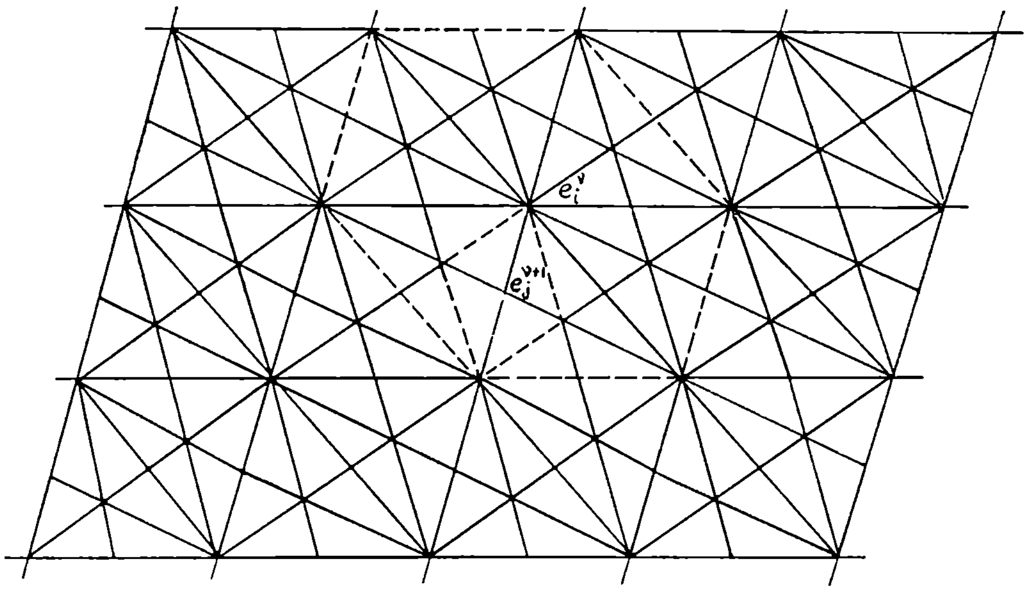


FIG. 148

## §3. The Δ-groups of open subsets of $S^n$. Formulation of the Alexander-Pontryagin duality and its fundamental consequences. Finite compacta

**§3.1. The group $\Delta^r(\Gamma)$, where $\Gamma$ is an open subset of $S^n$.** We have already considered (IV, 1.5, Example 2) the skeleton complex $K_\Gamma$ : the vertices of the complex $K_\Gamma$ are the points of the open set $\Gamma$; a set of vertices $\{e_0, e_1, \cdots, e_r\}$ forms a skeleton if the set of points $\{e_0, \cdots, e_r\}$ is contained in an open hemisphere of $S^n$ and if the closed convex hull of this set of vertices is contained in $\Gamma$. [All geometric terms (segment, convex set, simplex, barycentric subdivision of a simplex, etc.) used in this chapter are to be understood relative to the spherical metric of $S^n$.]

A simplex of the complex $K_\Gamma$ with skeleton $\{e_0, \cdots, e_r\}$ is the spherical simplex, perhaps degenerate, with vertices $e_0, \cdots, e_r$. With its simplexes defined in this way, the complex $K_\Gamma$ is an unrestricted simplicial complex.

*The $\Delta^r$-groups (homology groups) of the complex $K_\Gamma$ are called the $\Delta^r$-groups of the open subset $\Gamma$ and are denoted by $\Delta^r(\Gamma)$.*

Remark 1. We shall write $L^r(\Gamma)$, $Z^r(\Gamma)$, $H^r(\Gamma)$, etc. instead of $L^r(K_\Gamma)$, $Z^r(K_\Gamma)$, $H^r(K_\Gamma)$, etc. The chains of the complex $K_\Gamma$ are referred to simply as the *chains of the open set* $\Gamma$. If, in particular, $\Gamma = S^n$, we refer to the chains as chains of the space $S^n$.

Remark 2. Suppose $x^r \in L^r(\Gamma)$. The *simplexes of the chain* $x^r$ are, by definition, the simplexes on which $x^r$ is different from zero. The complex consisting of all the simplexes of $x^r$ and their faces is denoted by $| x^r |$ ; this complex is a finite unrestricted simplicial complex.

The *body of the chain* $x^r$ is defined to be the union of the closed convex hulls of all the simplexes of the complex $| x^r |$. The body of $x^r$ is a compact subset of $\Gamma$ and is denoted by $\bar{x}^r$.

**§3.2. The barycentric subdivision of a chain $x^r \in L^r(\Gamma)$.** Let $\Gamma$ be an open subset of $S^n \subset R^{n+1}$, and let $K$ be a finite closed subcomplex of $K_\Gamma$ of dimension $r$. In order to define the barycentric subdivision of $K$ we proceed as follows. We imbed the space $R^{n+1}$ in some $R^m$, $m \geq 2r + 1$, and transform the complex $K$ into an $r$-dimensional triangulation $K'$ contained in $R^m$ isomorphic to $K$ by means of a small displacement $S$ of the vertices of $K$. Denote by $K_1'$ the barycentric subdivision of the triangulation $K'$. All the simplexes of $K'$, and hence the simplexes of $K_1'$ contained in them, are mapped back into $S^n$ by the displacement $S^{-1}$ inverse to the displacement $S$; the image $K_1$ of the complex $K_1'$ under $S^{-1}$ is called the barycentric subdivision of $K$. Furthermore, the carrier of a vertex of $K_1$ in $K$ is, by definition, the simplex $T \subset K$ which is the image under $S^{-1}$ of the carrier of the corresponding vertex of $K_1'$.

Remark. It is clear that this definition of the barycentric subdivision of $K \subset K_\Gamma$ is not unique; however, all the barycentric subdivisions of $K$ are isomorphic.

We may now speak of barycentic subdivisions of $K$ of arbitrary order; for sufficiently large $h$, the skeletons of the barycentric subdivision of $K$ of order $h$ will have diameters less than a preassigned $\epsilon > 0$.

It follows that a sufficiently fine subdivision of a chain $x^r \in L^r(\Gamma)$ will yield a chain $s_h x^r$ whose simplexes are as small as desired. Furthermore, the term *normal displacement* in the sense of X, 3.1 has meaning, since the carriers in $K$ of the vertices of the barycentric subdivision $K_1$ of $K$ have been defined. The fundamental theorem on normal displacements (a normal displacement maps the barycentric subdivision $s_h x^r$ of order $h$ of a chain $x^r$ into $x^r$) thus remains valid. Hence the barycentric subdivision (of arbitrary order) of a cycle $z^r \in Z^r(\Gamma)$ is homologous to $z^r$ in $\Gamma$. Now suppose that $z^r \in Z^r(\Gamma)$ and that $P = \| Q_\alpha^n \|$ is a polyhedron contained in $\Gamma$ and containing $\bar{z}^r$ in its interior. (A polyhedron of this sort is most simply obtained by defining $Q_\alpha^n$ as the complex consisting of all the simplexes of a sufficiently fine simplicial decomposition of $S^n$ whose closures are contained in $\Gamma$.) Subjecting $z^r$ to a sufficiently fine barycentric subdivision and to a canonical displacement relative to the triangulation $Q_\alpha^n$, we obtain a cycle $z_\alpha^r$ of $Q_\alpha^n$ homologous to $z^r$ in $\Gamma$.

Hence,

3.21. *Every homology class* $\mathfrak{z}^r \in \Delta^r(\Gamma)$ *contains cycles of the triangulation* $Q_\alpha^n$, *where* $Q_\alpha^n$ *is an arbitrary triangulation whose body contains the body of every cycle* $z^r \in \mathfrak{z}^r$ *in its interior.*

It follows at once that:

3.22. *If* $r \geq n$, *then* $\Delta^r(\Gamma) = 0$.

### §3.3. The topological invariance of the groups $\Delta^r(\Gamma)$.

THEOREM 3.31. *If* $\Gamma_1$ *and* $\Gamma_2$ *are homeomorphic open subsets of* $S^n$, *then the groups* $\Delta^r(\Gamma_1)$ *and* $\Delta^r(\Gamma_2)$ *are isomorphic.*

The proof of Theorem 3.31 is preceded by the following definitions.

3.32. Every proper cycle of a compactum $\Phi \subset \Gamma$ is called a *proper cycle of the open subset* $\Gamma \subseteq S^n$.

Clearly, the proper $r$-cycles of an open subset $\Gamma \subseteq S^n$ with the usual definition of addition (see XI, 1.1) form a group which we denote by $\mathfrak{Z}^r(\Gamma)$.

3.33. A proper cycle $\mathfrak{z}^r \in \mathfrak{Z}^r(\Gamma)$ is said to be *homologous to zero* in $\Gamma$ if it is homologous to zero in a compactum contained in $\Gamma$.

It is clear that the proper $r$-cycles homologous to zero in $\Gamma$ form a subgroup $\mathfrak{H}^r(\Gamma)$ of $\mathfrak{Z}^r(\Gamma)$.

DEFINITION 3.34.

$$\Delta^r(\Gamma) = \mathfrak{Z}^r(\Gamma)/\mathfrak{H}^r(\Gamma). \tag{3.34}$$

Theorem 3.31 follows from

3.35. *The groups $\Delta^r(\Gamma)$ are isomorphic to the groups $\boldsymbol{\Delta}^r(\Gamma)$.*

*Proof of Theorem* 3.35. Let

$$\mathfrak{z}^r = (z_1^r, z_2^r, \cdots, z_h^r, \cdots)$$

be an arbitrary proper cycle of $\Gamma$. It is contained in a polyhedron $\Phi = \| K_\alpha^n \| \subset \Gamma$ and is mapped by a canonical displacement into a cycle $z_\alpha^r \in Z_\Delta^r(K_\alpha^n)$. This mapping induces a homomorphism $S$ of the group $\mathfrak{Z}^r(\Gamma)$ into $Z^r(\Gamma)$, which maps $\mathfrak{H}^r(\Gamma)$ into $H^r(\Gamma)$. The homomorphism $S$ induces a homomorphism (denoted by the same letter) of $\boldsymbol{\Delta}^r(\Gamma)$ into $\Delta^r(\Gamma)$. The homomorphism $S$ maps $\boldsymbol{\Delta}^r(\Gamma)$ *onto* $\Delta^r(\Gamma)$. Indeed, every homology class $\mathfrak{z}^r \in \Delta^r(\Gamma)$ contains a cycle $z_\alpha^r$ of a triangulation $Q_\alpha^n$ (see 3.21). Obviously, the homomorphism $S$ maps the proper cycle $(z_\alpha^r, z_{\alpha 1}^r, \cdots, z_{\alpha h}^r, \cdots)$, where $z_{\alpha h}^r$ is the barycentric subdivision of $z_\alpha^r$ of order $h$, into $z^r$. It remains to be proved that the homomorphism $S$ of $\boldsymbol{\Delta}^r(\Gamma)$ onto $\Delta^r(\Gamma)$ is an isomorphism. To show this it is sufficient to prove

3.351. *If $z^r = S\mathfrak{z}^r \in H^r(\Gamma)$, then $\mathfrak{z}^r \in \mathfrak{H}^r(\Gamma)$.*

To prove the last assertion, denote by $Q_\alpha^n$ the complex consisting of all the simplexes of a sufficiently fine simplicial subdivision of $S^n$ whose closure is contained in $\Gamma$. We may assume that

$$z^r = S\mathfrak{z}^r = z_\alpha^r \in H_\Delta^r(Q_\alpha^n),$$

that $\mathfrak{z}^r$ is a proper cycle of the polyhedron $Q_\alpha^n$ and that the displacement $S$ is effected in this polyhedron. Then 3.351 can be proved in exactly the same way as XI, Theorem 4.41.

**§3.4. The Alexander-Pontryagin duality and its fundamental consequences.** The fundamental theorem of this chapter may be formulated as follows:

THE ALEXANDER-PONTRYAGIN DUALITY. *Let $\Phi \subset S^n$ be a compactum*, and let $\Gamma = S^n \setminus \Phi$. *If $p$ and $q$ are two nonnegative integers whose sum is $n$, then the groups $\nabla^p(\Phi)$ and $\Delta^{q-1}(\Gamma)$ are isomorphic, where $\nabla^p(\Phi) = \nabla^{(0)}(\Phi)$ for $p = 0$ and $\Delta^{q-1}(\Gamma) = \Delta^{(0)}(\Gamma)$ [i.e., $\Delta^{(0)}(K_\Gamma)$] for $p = n - 1$.*

REMARK. In particular, the number of components of $\Gamma$ is $\pi^{n-1}(\Phi) + 1$, where $\pi^{n-1}(\Phi)$ is the rank of $\nabla^{n-1}(\Phi)$.

The Alexander-Pontryagin duality is one of the fundamental results of modern topology. Before we go on to prove it, we shall list some of its consequences and special cases. These will serve to illustrate its importance.

First, because homeomorphic compacta $\Phi$ have isomorphic groups $\nabla^r(\Phi)$, the Alexander-Pontryagin duality implies

3.41. *If $\Phi$ and $\Phi'$ are homeomorphic closed subsets of $S^n$, the groups $\Delta^r(S^n \setminus \Phi)$ and $\Delta^r(S^n \setminus \Phi')$ are isomorphic.*

In the same way, the Alexander-Pontryagin duality and Theorem 3.31 imply

3.42. *If* $\Gamma$ *and* $\Gamma'$ *are homeomorphic open subsets of* $R^n$, *then the groups* $\nabla^r(S^n \setminus \Gamma)$ *and* $\nabla^r(S^n \setminus \Gamma')$ *are isomorphic.*

A special case of Theorem 3.41 is

3.410. *Two homeomorphic closed subsets* $\Phi$ *and* $\Phi'$ *of* $S^n$ *separate* $S^n$ *into the same number of components.*

Hence (see II, Defs. 1.29 and 1.291),

3.43. *If one of two homeomorphic closed subsets of* $R^n$ *is an absolute (regular) boundary, then the other is also.*

Now suppose that $\Phi \subset S^n$ is a curved polyhedron. Then (IX, Theorem 3.51)

$$\nabla_0^p(\Phi) = \Theta^{p-1}(\Phi) + \Delta_{00}^p(\Phi) \tag{3.440}$$

has a finite number of generators. Therefore, $\Delta_0^{q-1}(\Gamma)$ also has a finite number of generators; consequently,

$$\Delta_0^{q-1}(\Gamma) = \Theta^{q-1}(\Gamma) + \Delta_{00}^{q-1}(\Gamma), \tag{3.441}$$

where $\Delta_{00}^{q-1}(\Gamma)$ is a free group and $\Theta^{q-1}(\Gamma)$ is the finite subgroup of $\Delta_0^{q-1}(\Gamma)$ consisting of all the elements of finite order of $\Delta_0^{q-1}(\Gamma)$. The group $\Theta^{q-1}(\Gamma)$ is called the $(q-1)$*st* torsion group of the open set $\Gamma$. The isomorphism $\nabla_0^p(\Phi) \approx \Delta_0^{q-1}(\Gamma)$ and (3.440), (3.441) imply that

$$\Theta^{p-1}(\Phi) \approx \Theta^{q-1}(\Gamma),$$

$$\Delta_{00}^p(\Phi) \approx \Delta_{00}^{q-1}(\Gamma).$$

Hence,

THE ALEXANDER DUALITY FOR CURVED POLYHEDRA IN $S^n$:

*If* $\Phi \subset S^n$ *is a curved polyhedron and* $p + q = n$, *then the pth Betti number of* $\Phi$ *is equal to the* $(q-1)$*st Betti number of the open set* $\Gamma = S^n \setminus \Phi$ *and the* $(p-1)$*st torsion group of* $\Phi$ *is isomorphic to the* $(q-1)$*st torsion group of* $\Gamma$.

Now let the coefficient domain be $J_m$. It is known [IX, (4.410)] that

$$\nabla_m^r(\Phi) \approx \Delta_m^r(\Phi)$$

for every curved polyhedron $\Phi$. Therefore:

THE ALEXANDER DUALITY (mod $m$) (for curved polyhedra $\Phi \subset S^n$). *The groups*

$$\Delta_m^p(\Phi) \text{ and } \Delta_m^{q-1}(\Gamma)$$

*are isomorphic.*

Finally, let $\Phi$ be an $(n-1)$-dimensional closed pseudomanifold in $S^n$. Then $\Delta_2^{n-1}(\Phi)$ is of order 2, so that $\Delta_2^{00}(\Gamma)$ is also of order 2, i.e., $\Gamma$ consists of two components. But then $\Delta_0^{00}(\Gamma)$ is infinite cyclic; hence $\Delta^{n-1}(\Phi)$ is also infinite cyclic and therefore $\Phi$ is orientable.

Since $\Delta^{n-1}(K_0) = 0$ for an arbitrary closed proper subcomplex $K_0$ of a triangulation $K^{n-1}$ of the pseudomanifold $\Phi$, no polyhedron of the form $\| K_0^{n-1} \|$, and hence, as is easily seen, no proper closed subset $\Phi_0 \subset \Phi$ separates $S^n$. In other words, $\Phi$ is a regular boundary (see II, Theorem 1.2). Therefore,

3.44. THE JORDAN-BROUWER THEOREM. *Every $(n - 1)$-dimensional closed pseudomanifold in $S^n$ is orientable, separates $S^n$ into precisely two domains and is the common boundary of these two domains.*

Since no closed proper subset of a closed $(n - 1)$-dimensional pseudomanifold contained in $S^n$ separates $S^n$, then in view of Theorem 3.410 the same is true of sets homeomorphic to proper subsets of such pseudomanifolds. It follows that no interior point of a compactum $\Phi \subset S^n$ has arbitrarily small neighborhoods (relative to $\Phi$) whose boundaries are homeomorphic to a proper subset of an $(n - 1)$-sphere. Since the boundary points of $\Phi$ obviously do have such neighborhoods, we have the following characterization of interior and boundary points of a compactum $\Phi \subset S^n$:

3.45. *A point $p$ is an interior point of a compactum $\Phi \subset S^n$ (relative to $S^n$) if, and only if, for all sufficiently small $\epsilon > 0$ no $\epsilon$-neighborhood of $p$ has a boundary homeomorphic to a proper subset of an $(n - 1)$-sphere. A point $q$ of $\Phi$ is a boundary point of $\Phi$ (relative to $S^n$) if, and only if, for every $\epsilon > 0$ there is an $\epsilon$-neighborhood of $q$ whose boundary is homeomorphic to a proper subset of an $(n - 1)$-sphere.*

Since these characterizations are topologically invariant, it follows that:

3.46. *A topological mapping of a compactum $\Phi \subset S^n$ into $S^n$ maps interior points of $\Phi$ (relative to $S^n$) into interior points, and boundary points of $\Phi$ (relative to $S^n$) into boundary points.*

REMARK. Since an interior point of an arbitrary subset $A \subseteq S^n$ is an interior point of some compactum $\Phi \subseteq A$, we have proved in full the theorem on the invariance of interior points, i.e., V, Theorem 3.1.

**§3.5. Special case: $\Phi$ is finite.** Suppose that $\Phi$ consists of a finite number of points $o_1, o_2, \cdots, o_\pi$. In view of the Remark at the end of 2.3, to prove the Alexander-Pontryagin duality in this special case it is necessary to prove the following propositions:

1°. *The group $\Delta^{n-1}(\Gamma, \mathfrak{A})$ is the direct sum of $\pi - 1$ groups isomorphic to $\mathfrak{A}$.*

2°. $\Delta^r(\Gamma, \mathfrak{A}) = 0$ *for* $r < n - 1$.

*Proof of* 1°. Let $K_1, K_2, \cdots, K_h, \cdots$ be a sequence of triangulations of $S^n$, each of which is a subdivision of the preceding, and such that mesh $K_h$ approaches zero with increasing $h$. With no loss in generality we may assume:

1) For every $h$ each $o_i$ is contained in an $n$-simplex of $K_h$, and no $n$-simplex of $K_h$ contains more than one of the points $o_i$.

2) The simplexes $T_{hi}$ of $K_h$ are indexed so that $o_i$ is contained in $T_{hi}$ for $i \leq \pi$.

We choose a definite orientation of $S^n$ and agree that all the $T_{hi}^n$ have the induced orientation. The result is a set of oriented simplexes $t_{hi}^n$. Finally, we set $z_{hi}^{n-1} = \Delta t_{hi}^n$, $i = 1, 2, \cdots, \pi$, and prove:

3.51. *Every cycle* $z^{n-1} \in Z^{n-1}(\Gamma, \mathfrak{A})$ *is homologous in* $\Gamma$ *to a linear combination of cycles* $z_{1i}^{n-1}$, $i = 1, 2, \cdots, \pi - 1$.

3.52. *If* $\sum_{i=1}^{\pi-1} c_i z_{1i}^{n-1} \sim 0$ *in* $\Gamma$, *then* $c_1 = \cdots = c_{\pi-1} = 0$.

In view of 3.35, it is legitimate to regard the cycles of $\Gamma$ as proper cycles, and we shall do so in the proof of these propositions. The actual proof is preceded by two remarks:

a) If $Q_h = K_h \setminus (T_{h1}^n \cup \cdots \cup T_{h\pi}^n)$, then

$$\sum_{i=1}^{\pi} z_{hi}^{n-1} \sim 0 \text{ in } Q_h \text{ (and hence also in } \Gamma).$$

For, since $\sum_i t_{hi}^n$ is a cycle, i.e.,

$$\Delta\left(\sum_{i\leq\pi} t_{hi}^n\right) + \Delta\left(\sum_{i>\pi} t_{hi}^n\right) = 0,$$

it follows that

$$\sum_{i=1}^{\pi} z_{hi}^{n-1} = \Delta\left(-\sum_{i>\pi} t_{hi}^n\right),$$

i.e.,

$$\sum_{i=1}^{\pi} z_{hi}^{n-1} \sim 0 \quad \text{in } Q_h .$$

b) For arbitrary $h$,

$$z_{hi}^{n-1} \sim z_{1i}^{n-1} \quad \text{in } \Gamma.$$

Indeed, a projection of the boundary of $T_{hi}^n$ onto the boundary of $T_{1i}^n$ from $o_i$ yields a deformation of $z_{hi}^{n-1}$ into $z_{1i}^{n-1}$; hence the two cycles are homotopic and therefore homologous in $\Gamma$.

In view of b), instead of 3.51 it is sufficient to prove

3.510. Every cycle $z^{n-1} \in Z^{n-1}(\Gamma, \mathfrak{A})$ is homologous in $\Gamma$ to a cycle of the form $\sum_{i=1}^{\pi-1} c_i z_{hi}^{n-1}$ for some $h$ depending on $z^{n-1}$.

We shall prove 3.510. By 3.21, $z^{n-1}$ is homologous in $K_h$ to a cycle $z_h^{n-1} \in Z^{n-1}(K_h, \mathfrak{A})$. The cycle $z_h^{n-1}$ is homologous to zero in $K_h$ : $z_h^{n-1} = \Delta x_h^n$. If $x_h^n = \sum_i a_i t_{hi}^n$, then

$$z_h^{n-1} - \sum_{i=1}^{\pi} a_i z_{hi}^{n-1} = \Delta\left(\sum_{i>\pi} a_i t_{hi}^n\right),$$

i.e.,

$$z_h^{n-1} \sim \sum_{i=1}^{\pi} a_i z_{hi}^{n-1} \quad \text{in } Q_h .$$

Since $a_\pi z_{h\pi}^{n-1} \sim -\sum_{i=1}^{\pi-1} a_\pi z_{hi}^{n-1}$ in $Q_h$ by a),

$$z_h^{n-1} \sim \sum_{i=1}^{\pi-1} (a_i - a_\pi) z_{hi}^{n-1} \quad \text{in } Q_h$$

and hence in $\Gamma$. This completes the proof.

*Proof of* 3.52. Suppose that

$$z^{n-1} = \sum_{i=1}^{\pi-1} c_i z_{1i}^{n-1} \sim 0 \quad \text{in } \Gamma.$$

Then there is an $h$ such that $z^{n-1} \sim 0$ in $\| Q_h \|$; hence (XI, 4.6), if $z_{hi}^{n-1}$ denotes the subdivision of $z_{1i}^{n-1}$ in $K_h$,

$$\sum_{i=1}^{\pi-1} c_i z_{hi}^{n-1} \sim 0 \quad \text{in } Q_h .$$

If $x_h^n = \sum_{i>\pi} a_i t_{hi}^n$ is a chain bounded by $\sum_{i=1}^{\pi-1} c_i z_{hi}^{n-1}$, then

$$\Delta\left(\sum_{i=1}^{\pi-1} c_i t_{hi}^n - \sum_{i>\pi} a_i t_{hi}^n\right) = 0.$$

Since $K_h$ is an $n$-dimensional pseudomanifold and $\sum_{i=1}^{\pi-1} c_i t_{hi}^n - \sum_{i>\pi} a_i t_{hi}^n$ is an $n$-cycle, the coefficients of all the $t_{hi}^n$ in this cycle must be the same. Since the coefficient of $t_{h\pi}^n$ is zero, it follows that all the $c_i$ (and all the $a_i$) are also zero. This completes the proof.

We denote by $\mathfrak{z}_i^{n-1}$ the element of $\Delta^{n-1}(\Gamma, \mathfrak{A})$ containing the cycle $z_{hi}^{n-1}$. It follows from 3.51 and 3.52 that every element of $\Delta^{n-1}(\Gamma, \mathfrak{A})$ is uniquely representable in the form

$$\mathfrak{z}^{n-1} = \sum c_i \mathfrak{z}_i^{n-1}, \qquad c_i \in \mathfrak{A} \qquad (i \leq \pi - 1),$$

i.e., that $\Delta^{n-1}(\Gamma, \mathfrak{A})$ is the direct sum of $\pi - 1$ groups each isomorphic to $\mathfrak{A}$. This proves 1°.

To prove 2°, suppose that $z^r \in Z^r(\Gamma, \mathfrak{A})$, $r < n - 1$, and that $z^r$ is a normal cycle for $r = 0$. Let $o'$ be a point of $\Gamma$ such that none of the planes spanned by $o'$ and any of the simplexes $z^r$ contains any of the points $o_1, \cdots, o_\pi$. The point $o'$ always exists, since these planes have dimension $\leq r + 1 \leq n - 1$.

Now deform the vertices of $z^r$ into $o'$ along the straight lines joining $o'$ to these vertices (the deformation may be pictured as a uniform flow of the vertices of $z^r$ into $o'$ which takes place in a unit of time along these straight lines). This is then a deformation in $\Gamma$ of $z^r$ into the $r$-cycle identically equal to zero (i.e., all its coefficients are zero). Hence every $r$-cycle of $\Gamma$ is, in our conditions, homotopic, and therefore homologous, to zero in $\Gamma$, which was to be proved.

This completes the proof of the Alexander-Pontryagin duality for a finite compactum.

## §4. The complexes $K_\alpha$, $K_{0\alpha}$, $K_\alpha^*$, $K_{0\alpha}^*$, $Q_{\alpha 1}$; the sets $G_\alpha$ and $\Gamma_\alpha$

(see IV, 5.4)

**§4.1. Definitions and notation** (see Fig. 149). In the rest of this chapter, $\Phi$ is an infinite closed subset of $S^n$ and $\Gamma = S^n \setminus \Phi$.

Let $K$ be a triangulation of $S^n$.

Let $k$ be a natural number; denote by $K_k$ the $k$th order barycentric subdivision of $K$, and by $K_{0k}$ the subcomplex of $K_k$ consisting of all the simplexes of $K_k$ containing points of $\Phi$ and all their faces. It is clear that

$$\Phi \subseteq \| K_{0k} \| , \tag{4.11}$$

$$\| K_{0.k+1} \| \subseteq \| K_{0k} \| \tag{4.12}$$

for all $k$. The body of the complex $K_k \setminus K_{0k}$ will be denoted by $G_k$.

Let $K_k^*$ be the complex of the barycentric stars of $K_k$, and let $K_{0k}^*$ be the subcomplex of $K_k^*$ consisting of the barycentric stars of $K_k^*$ dual to the simplexes of $K_{0k}$. Since $K_{0k}$ is a closed subcomplex of $K_k$, $K_{0k}^*$ is an open and $K_k^* \setminus K_{0k}^*$ is a closed subcomplex of $K_k^*$. Therefore, the body of $K_{0k}^*$ is open in $S^n$ and the body of $K_k^* \setminus K_{0k}^*$ is a polyhedron, one of whose triangulations is the *barycentric subdivision* $Q_{k1}$ (see IV, 5.4) of $K_k^* \setminus K_{0k}^*$. The set of all interior points relative to $S^n$ of the polyhedron $\| Q_{k1} \| = \| K_k^* \| \setminus \| K_{0k}^* \|$ will be denoted by $\Gamma_k$.

In view of IV, 5.42, we have

$$\Phi \subseteq \| K_{0k} \| \subseteq \| K_{0k}^* \| , \qquad \Gamma \supseteq \bar{G}_k \supset G_k \supset \| Q_{k1} \| \supset \Gamma_k . \tag{4.13}$$

Let $\epsilon_k$ be the mesh of $K_k$ (i.e., the maximum of the diameters of the simplexes of $K_k$). Then

$$\begin{gathered} \Phi \subset \| K_{0k}^* \| \subseteq S(\Phi, 2\epsilon_k), \\ \Gamma \supset \Gamma_k \supset S^n \setminus S(\Phi, 2\epsilon_k); \end{gathered} \tag{4.14}$$

since $\lim \epsilon_k = 0$,

$$\Gamma = \bigcup_{k=1}^{\infty} \Gamma_k . \tag{4.15}$$

Furthermore, since $\| K_{0k} \| \subset \| K_{0k}^* \|$ and $\| K_{0k}^* \|$ is open, $\rho(\Phi, \| Q_{k1} \| ) > 0$.

We now set $k_1 = 1$ and, assuming that $k_\alpha$ has been defined for the natural number $\alpha$, denote by $k_{\alpha+1}$ the first natural number satisfying the following conditions:

1°. $2\epsilon_{k_{\alpha+1}} < \rho(\Phi, \| Q_{k_\alpha 1} \| )$.

2°. $2\epsilon_{k_{\alpha+1}}$ is less than both $\epsilon_{k_\alpha}$ and the maximum of the diameters of the sets $\Phi \cap T_\alpha$, where $T_\alpha \in K_{k_\alpha}$.

Remark. If $O_\alpha$ is an open star of the triangulation $K_{k_\alpha}$, we see that because of 2° it cannot happen that *every* one of the sets $\Phi \cap O_\alpha$ is contained in some $\Phi \cap O_{\alpha+1}$ (i.e. the covering $\{\Phi \cap O_\alpha\}$ of $\Phi$ is not a refinement of the covering $\{\Phi \cap O_{\alpha+1}\}$); since, on the other hand, each set $\Phi \cap O_{\alpha+1}$ is contained in some $\Phi \cap O_\alpha$, 2° guarantees that the covering of $\Phi$ consisting of all the sets $\Phi \cap O_{\alpha+1}$ follows (in the sense of 1.3; see also I, 8.1) the covering

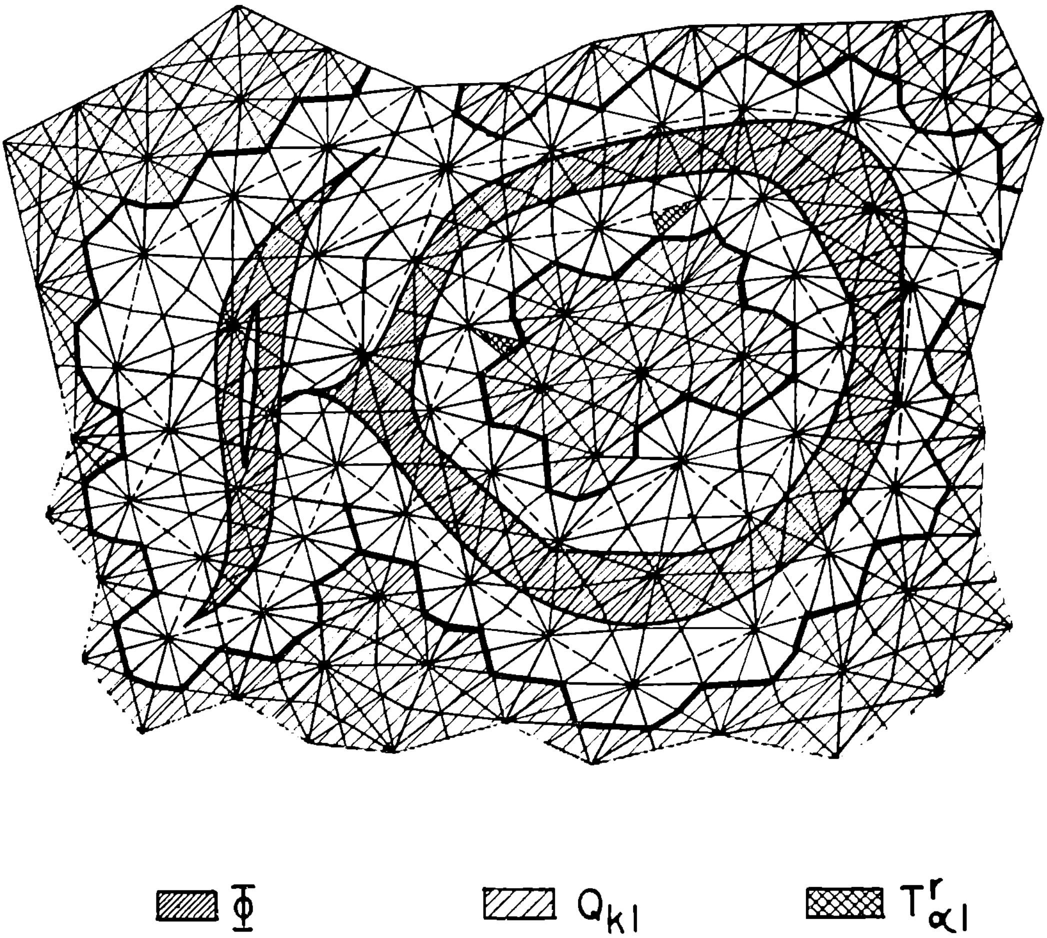


FIG. 149

consisting of all $\Phi \cap O_\alpha$. Condition 2° was introduced for precisely this reason.

Condition 1° implies that

$$\bar{S}(\Phi, 2\epsilon_{k_{\alpha+1}}) \subset S^n \setminus \| Q_{k_\alpha 1} \| ;$$

hence

$$\Gamma_{k_{\alpha+1}} \supset S^n \setminus \bar{S}(\Phi, 2\epsilon_{k_{\alpha+1}}) \supset \| Q_{k_\alpha 1} \| \supset \Gamma_{k_\alpha} .$$

For convenience, we now introduce the following notation. The complexes $K_{k_\alpha}$, $K_{0k_\alpha}$, $K_{k_\alpha}{}^*$, $K_{0k_\alpha}{}^*$, $Q_{k_\alpha 1}$ will be denoted by $K_\alpha$, $K_{0\alpha}$, $K_\alpha{}^*$, $K_{0\alpha}{}^*$, $Q_{\alpha 1}$, respectively; the open sets $\Gamma_{k_\alpha}$ and $G_{k_\alpha}$ will be written as $\Gamma_\alpha$ and $G_\alpha$. Finally, the barycentric subdivisions of the complexes $K_\alpha$, $K_{0\alpha}$ will be denoted by $K_{\alpha 1}$, $K_{0\alpha 1}$.

Because of what we have already proved in this subsection, we have

$$\Gamma = \bigcup_{\alpha=1}^{\infty} \| Q_{\alpha 1} \| = \bigcup_{\alpha=1}^{\infty} \Gamma_\alpha , \tag{4.16}$$

$$G_\alpha \supset \| Q_{\alpha 1} \| , \tag{4.17}$$

$$\Gamma_{\alpha+1} \supset \| Q_{\alpha 1} \| \supset \Gamma_\alpha . \tag{4.18}$$

**§4.2. The continuous mapping $C$ of $G_\alpha$ onto $\| Q_{\alpha 1} \|$.** We shall prove the following proposition:

4.21. *There exists a continuous mapping $C$ of $G_\alpha$ onto $\| Q_{\alpha 1} \|$, whose set of fixed points contains $\| Q_{\alpha 1} \|$.*

*Proof.* Let $T_{\alpha 1}^r$ be a simplex of $K_{\alpha 1}$, which does not belong to $Q_{\alpha 1}$, but is contained in a simplex

$$T_\alpha^{n_0} \in K_\alpha \setminus K_{0\alpha} .$$

Then

$$T_{\alpha 1}^r = (T_\alpha^{n_0} > \cdots > T_\alpha^{n_r}),$$

where $T_\alpha^{n_0} \in K_\alpha \setminus K_{0\alpha}$, but $T_\alpha^{n_r} \in K_{0\alpha}$. Let $T_\alpha^{n_{i+1}}$ be the first of the simplexes $T_\alpha^{n_0}, \cdots, T_\alpha^{n_r}$ contained in $K_{0\alpha}$. Since $K_{0\alpha}$ is a closed complex, we have

$$T_\alpha^{n_{i+1}} \in K_{0\alpha}, \cdots, T_\alpha^{n_i} \in K_{0\alpha}$$

and

$$T_{\alpha 1}' = (T_\alpha^{n_0} > \cdots > T_\alpha^{n_i}) \in Q_{\alpha 1},$$

$$T_{\alpha 1}'' = (T_\alpha^{n_{i+1}} > \cdots > T_\alpha^{n_r}) \in K_{0\alpha 1} .$$

Every point $p$ of the simplex $T_{\alpha 1}^r$ uniquely determines the segment $p''p'$ containing $p$ and joining a point $p'' \in T_{\alpha 1}''$ with a point $p' \in T_{\alpha 1}'$. Set $C(p) = p'$. The mapping $C$ is defined for all $p \in G_\alpha \setminus \| Q_{\alpha 1} \|$. If $p \in \| Q_{\alpha 1} \|$, set $C(p) = p$. The mapping $C$ is continuous and satisfies the requirements of Theorem 4.21.

**§4.3. Some properties of the group $Z^r(\Gamma)$.** It follows from (4.16) and (4.18) that every chain $x$ of $\Gamma$ is a chain of $\Gamma_\alpha$, and hence of $\| Q_{\alpha 1} \|$, for every sufficiently large $\alpha$. Consequently, in view of 3.21, every cycle $z \in Z^r(\Gamma)$ is homologous to a cycle $z_{\alpha 1} \in Z^r(Q_{\alpha 1})$ in $\Gamma$.

But $Q_{\alpha 1}$ is the barycentric subdivision of $K_\alpha^* \setminus K_{0\alpha}^* \subseteq K_\alpha^*$; hence, because of XIII, Theorem 2.11, $z_{\alpha 1} \in Z^r(Q_{\alpha 1})$ is homologous in $Q_{\alpha 1}$, and therefore in $\Gamma$, to a cycle of the form $s_{\alpha 1}^\alpha z_\alpha^*$, where $z_\alpha^* \in Z_\Delta^r(K_\alpha^* \setminus K_{0\alpha}^*)$.

Therefore,

4.31. *Every homology class $\mathfrak{z}^r \in \Delta^r(\Gamma)$ contains, for every sufficiently large $\alpha$, a cycle of the form $s_{\alpha 1}^\alpha z_\alpha^*$, where*

$$z_\alpha^* \in Z_\Delta^r(K_\alpha^* \setminus K_{0\alpha}^*).$$

Furthermore, XIII, Theorem 2.12 implies

4.32. *If $s_{\alpha 1}^\alpha z_\alpha^* \in H_\Delta^r(Q_{\alpha 1})$, $z_\alpha^* \in Z_\Delta^r(K_\alpha^* \setminus K_{0\alpha}^*)$, then*

$$z_\alpha^* \in H_\Delta^r(K_\alpha^* \setminus K_{0\alpha}^*).$$

## §5. A combinatorial lemma

**§5.1. Introductory remarks and formulation of the lemma.** Since $K_{\alpha 1}$ is the barycentric subdivision of $K_\alpha$, every simplex $T_{\alpha 1} \in K_{\alpha 1}$ is contained in exactly one barycentric star of $K_\alpha$. We shall call this star the *carrier* of the simplex $T_{\alpha 1}$ in $K_\alpha^*$; a simplex $T_{\alpha 1} \in K_{\alpha 1}$ is said to be a *principal simplex* if its dimension is the same as the dimension of its carrier in $K_\alpha^*$.

It is easily verified that a principal simplex $T_{\alpha 1}^q \in K_{\alpha 1}$ is of the form

$$T_{\alpha 1}^q = (T_\alpha^n > T_\alpha^{n-1} > \cdots > T_\alpha^p), \tag{5.11}$$

where $T_\alpha^n \in K_\alpha, \cdots, T_\alpha^p \in K_\alpha$ are the simplexes whose centroids are the vertices of $T_{\alpha 1}^q$.

To show this, suppose that $*T_\alpha^q \in K_\alpha^*$ is the carrier of the principal simplex $T_{\alpha 1}^q$. The barycentric star $*T_\alpha^q$ is dual to a simplex $T_\alpha^p \in K_\alpha$, and, by definition, the $q$-simplexes of $*T_\alpha^q$ have the form (5.11).

If $T_\alpha^p \in K_\alpha$ and $*T_\alpha^q \in K_\alpha^*$ are dual, we shall say for convenience that the simplex $T_\alpha^p$ is dual to each of the principal simplexes $T_{\alpha 1}^q \in *T_\alpha^q$.

Let $K_\alpha$ and $K_\beta$, $\beta > \alpha$, be two of the complexes $K_1, K_2, \cdots, K_\alpha, \cdots$ (see 4.2), and let $S_\alpha^\beta$ be a normal simplicial mapping (see X, 3.1) of $K_\beta$ into $K_\alpha$. We denote by $s_{\beta 1}^{\cdots}$ the subdivision operator in $K_{\beta 1}$ (i.e., the operator which maps the chains of any $K_{\alpha'}$ or $K_{\alpha'}^*$, $\alpha' \leq \beta$, into their subdivisions in $K_{\beta 1}$); $D_\alpha^q$ will stand for the isomorphism of the group $L^p(K_\alpha)$ onto the group $L^q(K_\alpha^*)$ defined in XIII, 3.3. Then

FUNDAMENTAL LEMMA 5.1. *The relation*

$$s_{\beta 1}^{\beta} D_\beta^q \bar{S}_\beta^\alpha z_\alpha^p \sim s_{\beta 1}^{\alpha} D_\alpha^q z_\alpha^p \quad \text{in } Q_{\beta 1} \tag{5.1}$$

*holds for every cocyle* $z_\alpha^p \in Z_\nabla^p(K_\alpha)$.

The proof of Lemma 5.1 is quite long and will not be complete till the end of this section. We begin by assigning a definite orientation to $S^n$; $t_\alpha^n$ and $t_\beta^n$ will denote the orientations of $T_\alpha^n \in K_\alpha$, $T_\beta^n \in K_\beta$ coherent with that of $S^n$.

Furthermore, for $p < n$, we choose arbitrary, but definite, orientations $t_\alpha^p$ of each simplex $T_\alpha^p \in K_\alpha$. Having assigned the orientations $t_\alpha^p$, we choose the orientations $t_\beta^p$ of every $T_\beta^p \in K_\beta$ so that if $S_\alpha^\beta T_\beta^p = T_\alpha^p$, then $S_\alpha^\beta t_\beta^p = t_\alpha^p$; if the dimension of $S_\alpha^\beta T_\beta^p$ is less than $p$, the orientation $t_\beta^p$ is chosen arbitrarily.

The orientations $*t_\alpha^q$, $*t_\beta^q$ of the barycentric stars $*T_\alpha^q \in K_\alpha^*$ and $*T_q^\beta \in K_\beta^*$ dual to the corresponding simplexes $T_\alpha^p \in K_\alpha$ and $T_\beta^p \in K_\beta$ are chosen so that

$$(t_\alpha^p \times *t_\alpha^q) = 1, \qquad (t_\beta^p \times *t_\beta^q) = 1.$$

The corresponding orientations of the principal simplexes $T_{\alpha 1}^q \in *T_\alpha^q$ and $T_{\beta 1}^q \in *T_\beta^q$ are denoted by $t_{\alpha 1}^q$, $t_{\beta 1}^q$.

**§5.2. The fundamental identity.** We define an auxiliary simplicial mapping $S_{\alpha 1}{}^{\beta 1}$ of $K_{\beta 1}$ into $K_{\alpha 1}$ as follows: to a vertex of the complex $K_{\beta 1}$, which is the centroid of a simplex $T_\beta \subset K_\beta$, assign the vertex of $K_{\alpha 1}$ which is the centroid of the simplex $S_\alpha{}^\beta T_\beta$. This mapping assigns to a simplex

$$T_{\beta 1}{}^r = (T_\beta{}^{n_0} > T_\beta{}^{n_1} > \cdots > T_\beta{}^{n_r}) \in K_{\beta 1}$$

the simplex

$$(5.21) \quad S_{\alpha 1}{}^{\beta 1} T_{\beta 1}{}^r = T_{\alpha 1} = (S_\alpha{}^\beta T_\beta{}^{n_0} \geq \cdots \geq S_\alpha{}^\beta T_\beta{}^{n_r}) \subset K_{\alpha 1}.$$

We now investigate the effect of $S_{\alpha 1}{}^{\beta 1}$ on the principal simplexes of $K_{\beta 1}$.

5.22. *Every q-simplex*

$$T_{\alpha 1}{}^q = (T_\alpha{}^{n_0} > T_\alpha{}^{n_1} > \cdots > T_\alpha{}^{n_q}),$$

*which is not a principal simplex, satisfies the condition* $n_q < p$.

For, since $n_i - n_{i+1} \geq 1$, $n \geq n_0$ $(0 \leq i \leq q = n - p)$, then $n_q \geq p$ would imply that $n_0 = n$, $n_1 = n - 1, \cdots, n_q = p$, i.e., it would imply that $T_{\alpha 1}{}^q$ is a principal simplex. This proves 5.22.

If we now ask which principal simplexes

$$(5.23_\beta) \qquad T_{\beta 1}{}^q = (T_\beta{}^n > \cdots > T_\beta{}^p) \in K_{\beta 1}$$

are mapped by $S_{\alpha 1}{}^{\beta 1}$ onto a given principal simplex

$$(5.23_\alpha) \qquad T_{\alpha 1}{}^q = (T_\alpha{}^n > \cdots > T_\alpha{}^p) \in K_{\alpha 1},$$

we find that if $S_{\alpha 1}{}^{\beta 1} T_{\beta 1}{}^q = T_{\alpha 1}{}^q$, then $S_\alpha{}^\beta T_\beta{}^n = T_\alpha{}^n$. On the other hand, if $S_\alpha{}^\beta T_\beta{}^n = T_\alpha{}^n$, then $S_{\alpha 1}{}^{\beta 1}$ maps precisely one simplex of the form $(5.23_\beta)$ onto the given simplex $(5.23_\alpha)$.

In fact, if $S_{\alpha 1}{}^{\beta 1} T_{\beta 1}{}^q = T_{\alpha 1}{}^q$, where $T_{\beta 1}{}^q$, $T_{\alpha 1}{}^q$ are of the form $(5.23_\beta)$ and $(5.23_\alpha)$, then each of the simplexes $T_\beta{}^{n-1}, \cdots, T_\beta{}^p$ is uniquely determined as the unique face of $T_\beta{}^n$ mapped by $S_\alpha{}^\beta$ onto $T_\alpha{}^{n-1}, \cdots, T_\alpha{}^p$, respectively.

Hence, the principal simplexes $T_{\beta 1}{}^q \in K_{\beta 1}$ mapped by $S_{\alpha 1}{}^{\beta 1}$ onto a given principal simplex $(5.23_\alpha)$ correspond (1–1) to the simplexes $T_\beta{}^n \in K_\beta$ mapped by $S_\alpha{}^\beta$ onto $T_\alpha{}^n$.

REMARK. Denote by $\tilde{S}_\alpha{}^\beta$ the (nondegenerate) affine mapping of the closed simplex $\bar{T}_\beta{}^n$ onto $\bar{T}_\alpha{}^n$ induced by the mapping $S_\alpha{}^\beta$ of the vertices of $T_\beta{}^n$ onto those of $T_\alpha{}^n$. Since the nondegenerate affine mapping $\tilde{S}_\alpha{}^\beta$ of the closed simplex $\bar{T}_\beta{}^n$ onto $\bar{T}_\alpha{}^n$ maps the centroid of each face of $T_\beta{}^n$ into the centroid of its image under $S_\alpha{}^\beta$, then

$$S_{\alpha 1}{}^{\beta 1} T_{\beta 1}{}^q = \tilde{S}_\alpha{}^\beta T_\beta{}^q$$

for every principal simplex $T_{\beta 1}{}^q \in K_{\beta 1}$. We shall use this remark in what follows.

Suppose that $T_\alpha{}^p \in K_\alpha$, $T_\beta{}^p \in K_\beta$ are the simplexes dual to $T_{\alpha 1}{}^q$, $T_{\beta 1}{}^q$, respectively. Passing to the oriented simplexes $t_\alpha^p$, $t_\alpha^n$, $t_\beta^p$, $t_\beta^n$, $t_{\alpha1}^q$, $t_{\beta1}^q$, let

$$S_\alpha^\beta t_\beta^n = \zeta t_\alpha^n, \qquad \zeta = \pm 1.$$

Then

$$\tilde{S}_\alpha^\beta t_{\beta1}^q = S_{\alpha1}^{\beta1} t_{\beta1}^q = \eta t_{\alpha1}^q.$$

We shall prove that $\eta = \zeta$.

To this end we note first that in view of VII, Theorem 2.3,

$$(\tilde{S}_\alpha^\beta t_\beta^p \times \tilde{S}_\alpha^\beta t_{\beta1}^q) = \zeta(t_\beta^p \times t_{\beta1}^q), \tag{5.23}$$

*where the intersection number of the simplexes is taken to be the intersection number of the carrying planes of the simplexes, oriented coherently with the simplexes.* (This agreement violates the convention of VII, 2.2 since the simplexes we are now considering are not in general position; therefore, this agreement is to be binding only in the course of this proof.) In view of the choice of the orientation $t_\beta^p$, we have

$$\tilde{S}_\alpha^\beta t_\beta^p = S_\alpha^\beta t_\beta^p = t_\alpha^p.$$

Moreover,

$$(t_\beta^p \times t_{\beta1}^q) = 1, \qquad (t_\alpha^p \times t_{\alpha1}^q) = 1.$$

Hence, the left side of (5.23) yields

$$(\tilde{S}_\alpha^\beta t_\beta^p \times \tilde{S}_\alpha^\beta t_{\beta1}^q) = (t_\alpha^p \times \eta t_{\alpha1}^q) = \eta(t_\alpha^p \times t_{\alpha1}^q) = \eta,$$

while the right side of (5.23) is

$$\zeta(t_\beta^p \times t_{\beta1}^q) = \zeta \cdot 1 = \zeta.$$

Therefore, $\eta = \zeta$, and the coefficient of $t_{\alpha1}^q$ in $S_{\alpha1}^{\beta1} \sum t_{\beta1}^q$ (summed over all principal $t_{\beta1}^q$) is equal to the coefficient of $t_\alpha^n$ in $S_\alpha^\beta \sum t_\beta^n$ (summed over all $t_\beta^n$). But the latter coefficient is equal to 1 [see X, (3.11)].

We have therefore proved

5.24. *The coefficient of a given principal oriented simplex $t_{\alpha1}^q$ of $K_\alpha$ in $S_{\alpha1}^{\beta1} \sum t_{\beta1}^q$ is equal to* 1; *i.e.,*

$$S_{\alpha1}^{\beta1} \sum t_{\beta1}^q = \sum t_{\alpha1}^q \tag{5.24}$$

(*summed over all principal $q$-simplexes of $K_{\beta1}$ and $K_{\alpha1}$, respectively*).

We shall now prove the following fundamental relation, from which we shall later also derive Lemma 5.1.

5.2. *The identity*

$$S_{\alpha1}^{\beta1} s_{\beta1}^{\beta} D_\beta^q \tilde{S}_\beta^\alpha x_\alpha^p = s_{\alpha1}^{\alpha} D_\alpha^q x_\alpha^p \tag{5.2}$$

*holds for an arbitrary chain $x_\alpha^p \in L^p(K_\alpha)$.*

*Proof.* The relations

$$(s_{\beta 1}{}^{\beta} D_{\beta}{}^{q} \bar{S}_{\beta}{}^{\alpha} x_{\alpha}{}^{p} \cdot t_{\beta 1}{}^{q}) = (D_{\beta}{}^{q} \bar{S}_{\beta}{}^{\alpha} x_{\alpha}{}^{p} \cdot {}^{*}t_{\beta}{}^{q}) = (\bar{S}_{\beta}{}^{\alpha} x_{\alpha}{}^{p} \cdot t_{\beta}{}^{p})$$
$$= (x_{\alpha}{}^{p} \cdot S_{\alpha}{}^{\beta} t_{\beta}{}^{p})$$

hold for an arbitrary principal simplex $t_{\beta 1}{}^{q}$ of $K_{\beta 1}$, where ${}^{*}t_{\beta}{}^{q}$ is the carrier of $t_{\beta 1}{}^{q}$ in $K_{\beta}{}^{*}$ and $t_{\beta}{}^{p}$ is the simplex dual to the star ${}^{*}t_{\beta}{}^{q}$. For brevity, we set

$$y_{\beta 1}{}^{q} = s_{\beta 1}{}^{\beta} D_{\beta}{}^{q} \bar{S}_{\beta}{}^{\alpha} x_{\alpha}{}^{p}.$$

We have just found that

$$(5.25) \qquad (y_{\beta 1}{}^{q} \cdot t_{\beta 1}{}^{q}) = (x_{\alpha}{}^{p} \cdot S_{\alpha}{}^{\beta} t_{\beta}{}^{p})$$

for an arbitrary principal $T_{\beta 1}{}^{q} \in K_{\beta 1}$. For a nonprincipal simplex $T_{\beta 1}{}^{q} \in K_{\beta 1}$, we have $(y_{\beta 1}{}^{q} \cdot t_{\beta 1}{}^{q}) = 0$ (for arbitrary orientation $t_{\beta 1}{}^{q}$ of $T_{\beta 1}{}^{q}$), because of the definition of the operator $s_{\beta 1}{}^{\beta}$.

Now let

$$(5.26) \qquad T_{\alpha 1}{}^{q} = (T_{\alpha}{}^{n_0} > T_{\alpha}{}^{n_1} > \cdots > T_{\alpha}{}^{n_q})$$

be an arbitrary $q$-simplex of $K_{\alpha 1}$; if $T_{\alpha 1}{}^{q}$ is a principal simplex, then the orientation $t_{\alpha 1}{}^{q}$ of $T_{\alpha 1}{}^{q}$ has already been prescribed; if $T_{\alpha 1}{}^{q}$ is a nonprincipal simplex, we denote by $t_{\alpha 1}{}^{q}$ any orientation of $T_{\alpha 1}{}^{q}$. Let

$$T_{\beta 1 . j}{}^{q} = (T_{\beta . j}{}^{n} > T_{\beta . j}{}^{n-1} > \cdots > T_{\beta . j}{}^{p}) \qquad (1 \leq j \leq \nu)$$

be all the principal simplexes of $K_{\beta 1}$ mapped by $S_{\alpha 1}{}^{\beta 1}$ onto $T_{\alpha 1}{}^{q}$ (of course, distinct simplexes $T_{\beta 1 , j}{}^{q}$ may be dual to the same simplex $T_{\beta , j}{}^{p}$, so that the simplexes $T_{\beta , j}{}^{p}$ are not necessarily distinct for distinct values of $j$).

Let $S_{\alpha 1}{}^{\beta 1} t_{\beta 1 . j}{}^{q} = \eta_j t_{\alpha 1}{}^{q}$, where $\eta_j = \pm 1$.

Because of the definition of the operator $S_{\alpha 1}{}^{\beta 1}$ and (5.25), we have

$$(5.27) \quad (S_{\alpha 1}{}^{\beta 1} y_{\beta 1}{}^{q} \cdot t_{\alpha 1}{}^{q}) = \textstyle\sum_{j=1}^{\nu} (y_{\beta 1}{}^{q} \cdot \eta_j t_{\beta 1 , j}{}^{q}) = \sum_{j=1}^{\nu} \eta_j (x_{\alpha}{}^{p} \cdot S_{\alpha}{}^{\beta} t_{\beta , j}{}^{p}).$$

We consider two cases:

1°. $T_{\alpha 1}{}^{q}$ is not a principal simplex. Then by 5.22 and (5.26), $n_q < p$; since $S_{\alpha}{}^{\beta} T_{\beta , j}{}^{p} = T_{\alpha}{}^{n_q}$, $(x_{\alpha}{}^{p} \cdot S_{\alpha}{}^{\beta} t_{\beta , j}{}^{p}) = 0$ for all $j = 1, 2, \cdots, \nu$. In conjunction with (5.27) this yields

$$(S_{\alpha 1}{}^{\beta 1} y_{\beta 1}{}^{q} \cdot t_{\alpha 1}{}^{q}) = 0.$$

In view of the definition of $s_{\alpha 1}{}^{\alpha}$, the value of the chain $s_{\alpha 1}{}^{\alpha} D_{\alpha}{}^{q} x_{\alpha}{}^{p}$ on a nonprincipal simplex $t_{\alpha 1}{}^{q}$ is also zero, so that

$$(S_{\alpha 1}{}^{\beta 1} y_{\beta 1}{}^{q} \cdot t_{\alpha 1}{}^{q}) = (s_{\alpha 1}{}^{\alpha} D_{\alpha}{}^{q} x_{\alpha}{}^{p} \cdot t_{\alpha 1}{}^{q})$$

for a nonprincipal $T_{\alpha 1}{}^{q}$.

2°. $T_{\alpha 1}{}^{q}$ is a principal simplex. Then (5.26) becomes

$$T_{\alpha 1}{}^{q} = (T_{\alpha}{}^{n} > T_{\alpha}{}^{n-1} > \cdots > T_{\alpha}{}^{p}),$$

and

$$S_\alpha^\beta T_{\beta,j}^p = T_\alpha^p;$$

for our choice of orientation $t_{\beta,j}^p$ it is also true that

$$S_\alpha^\beta t_{\beta,j}^p = t_\alpha^p.$$

Inserting this in (5.27), we obtain

$$(S_{\alpha 1}^{\beta 1} y_{\beta 1}^q \cdot t_{\alpha 1}^q) = \sum_j \eta_j (x_\alpha^p \cdot t_\alpha^p) = (x_\alpha^p \cdot t_\alpha^p) \sum_j \eta_j .$$

In view of 5.24, $\sum_j \eta_j = 1$, so that

$$(S_{\alpha 1}^{\beta 1} y_{\beta 1}^q \cdot t_{\alpha 1}^q) = (x_\alpha^p \cdot t_\alpha^p).$$

On the other hand,

$$(s_{\alpha 1}^\alpha D_\alpha^q x_\alpha^p \cdot t_{\alpha 1}^q) = (D_\alpha^q x_\alpha^p \cdot {}^*t_\alpha^q) = (x_\alpha^p \cdot t_\alpha^p),$$

i.e.,

$$(S_{\alpha 1}^{\beta 1} y_{\beta 1}^q \cdot t_{\alpha 1}^q) = (s_{\alpha 1}^\alpha D_\alpha^q x_\alpha^p \cdot t_{\alpha 1}^q).$$

Hence, both sides of (5.2) assume the same value on an arbitrary $q$-simplex of $K_{\alpha 1}$, q.e.d.

§**5.3. The completion of the proof of Lemma 5.1** is based on the following proposition:

5.31. If a cocycle $z_\alpha^p \in Z_\nabla^p(K_\alpha)$ is equal to zero on $K_{0\alpha}$ and $z_{\beta 1}^q = s_{\beta 1}^\beta D_\beta^q \bar{S}_\beta^\alpha z_\alpha^p$ is different from zero on a given $T_{\beta 1}^q \in K_{\beta 1}$, then both of the closed simplexes $\bar{T}_{\beta 1}^q$ and $\bar{S}_{\alpha 1}^{\beta 1} T_{\beta 1}^q$ (where $\bar{S}_{\alpha 1}^{\beta 1} T_{\beta 1}^q$ denotes the closure of $S_{\alpha 1}^{\beta 1} T_{\beta 1}^q$) are contained in a convex subset of $G_\alpha = \| K_\alpha \| \setminus \| K_{0\alpha} \|$, namely in a set $O(| T_\alpha^n |; T_\alpha^p)$, where $T_\alpha^p \in K_\alpha \setminus K_{0\alpha}$ and $T_\alpha^n > T_\alpha^p$. [Here and in the sequel, wherever necessary for convenience in notation, $O(| T_\alpha^n |; T_\alpha^p)$ denotes the star of $T_\alpha^p$ in the combinatorial closure $| T_\alpha^n |$ of $T_\alpha^n$.]

REMARK. If $T_\alpha^p < T_\alpha^n$, then $O(| T_\alpha^n |; T_\alpha^p)$ consists of $T_\alpha^p$ and all the faces of $T_\alpha^n$ (including $T_\alpha^n$ itself) which have $T_\alpha^p$ as a face; among the points of the closed simplex $\bar{T}_\alpha^n$, the points of $O(| T_\alpha^n |; T_\alpha^p)$ are characterized as those all of whose barycentric coordinates corresponding to the vertices of $T_\alpha^p$ are positive. This property implies that $O(| T_\alpha^n |; T_\alpha^p)$ is convex.

To prove 5.31, we note that according to the conditions imposed, $T_{\beta 1}^q$ is a principal simplex:

$$T_{\beta 1}^q = (T_\beta^n > \cdots > T_\beta^{n-i} > \cdots > T_\beta^p),$$

and that $\bar{S}_\beta^\alpha z_\alpha^p$ does not vanish on $T_\beta^p$. Therefore,

$$S_\alpha^\beta T_\beta^p = T_\alpha^p \in K_\alpha \setminus K_{0\alpha} .$$

The simplex $S_{\alpha 1}{}^{\beta 1} T_{\beta 1}{}^{q}$ has the form

$$S_{\alpha 1}{}^{\beta 1} T_{\beta 1}{}^{q} = (S_{\alpha}{}^{\beta} T_{\beta}{}^{n} \geq \cdots \geq S_{\alpha}{}^{\beta} T_{\beta}{}^{p}),$$

where $S_{\alpha}{}^{\beta} T_{\beta}{}^{p} = T_{\alpha}{}^{p}$.

Let $T_{\alpha}{}^{n}$ be the carrier of $T_{\beta}{}^{n}$ in $K_{\alpha}$, and set $T_{\alpha}{}^{n0} = S_{\alpha}{}^{\beta} T_{\beta}{}^{n}$. Since $S_{\alpha}{}^{\beta}$ is a normal mapping, $T_{\alpha}{}^{n} \geq T_{\alpha}{}^{n0}$, so that

$$O(|\, T_{\alpha}{}^{n0} \,|;\, T_{\alpha}{}^{p}) \subseteq O(|\, T_{\alpha}{}^{n} \,|;\, T_{\alpha}{}^{p}).$$

The vertices of $S_{\alpha 1}{}^{\beta 1} T_{\beta 1}{}^{q}$ are the centroids of the simplexes

$$T_{\alpha}{}^{n0} = S_{\alpha}{}^{\beta} T_{\beta}{}^{n} \geq \cdots \geq S_{\alpha}{}^{\beta} T_{\beta}{}^{p} = T_{\alpha}{}^{p},$$

and are therefore contained in the convex set

$$O(|\, T_{\alpha}{}^{n0} \,|;\, T_{\alpha}{}^{p}) \subseteq O(|\, T_{\alpha}{}^{n} \,|;\, T_{\alpha}{}^{p});$$

hence the closed simplex $\bar{S}_{\alpha 1}{}^{\beta 1} T_{\beta 1}{}^{q}$ is also contained in the set $O(|\, T_{\alpha}{}^{n} \,|;\, T_{\alpha}{}^{p})$:

$$\bar{S}_{\alpha 1}{}^{\beta 1} T_{\beta 1}{}^{q} \subseteq O(|\, T_{\alpha}{}^{n} \,|;\, T_{\alpha}{}^{p}) \tag{5.311}$$

(where $\bar{S}_{\alpha 1}{}^{\beta 1} T_{\beta 1}{}^{q}$ denotes the closure of $S_{\alpha 1}{}^{\beta 1} T_{\beta 1}{}^{q}$).

It remains to be proved that

$$\bar{T}_{\beta 1}{}^{q} \subset O(|\, T_{\alpha}{}^{n} \,|;\, T_{\alpha}{}^{p}). \tag{5.312}$$

Letting $T_{\alpha}{}^{ni}$ $(1 \leq i \leq q)$ stand for the carrier of the simplex $T_{\beta}{}^{n-i}$ in $K_{\alpha}$ (and recalling that $T_{\alpha}{}^{n}$ is the carrier of $T_{\beta}{}^{n}$), we see that

$$T_{\alpha}{}^{n} \geq T_{\alpha}{}^{n1} \geq \cdots \geq T_{\alpha}{}^{nq}. \tag{5.313}$$

Since $S_{\alpha}{}^{\beta}$ is a normal mapping, $T_{\alpha}{}^{p} = S_{\alpha}{}^{\beta} T_{\beta}{}^{p}$ is a face of $T_{\alpha}{}^{nq}$; hence, by (5.313),

$$T_{\alpha}{}^{n} \geq T_{\alpha}{}^{n1} \geq \cdots \geq T_{\alpha}{}^{nq} \geq T_{\alpha}{}^{p}. \tag{5.314}$$

It follows that all the simplexes of (5.314) are contained in $O(|\, T_{\alpha}{}^{n} \,|;\, T_{\alpha}{}^{p})$; in particular, $O(|\, T_{\alpha}{}^{n} \,|;\, T_{\alpha}{}^{p})$ contains the centroids of the simplexes

$$T_{\beta}{}^{n} \subseteq T_{\alpha}{}^{n},\ T_{\beta}{}^{n-1} \subseteq T_{\alpha}{}^{n1},\ \cdots,\ T_{\beta}{}^{p} \subseteq T_{\alpha}{}^{nq},$$

i.e., the vertices of $T_{\beta 1}{}^{q}$. Since $O(|\, T_{\alpha}{}^{n} \,|;\, T_{\alpha}{}^{p})$ is a convex set, and contains all the vertices of $T_{\beta 1}{}^{q}$, it also contains the closed simplex $\bar{T}_{\beta 1}{}^{q}$. This proves 5.31.

We shall now complete the proof of Lemma 5.1. In view of 5.31, all the skeletons of the prisms spanned by $|\, T_{\beta 1}{}^{q} \,|$ and $S_{\alpha 1}{}^{\beta 1} |\, T_{\beta 1}{}^{q} \,|$ are contained in the convex set

$$O(|\, T_{\alpha}{}^{n} \,|;\, T_{\alpha}{}^{p}) \subseteq \|\, K_{\alpha} \,\| \setminus \|\, K_{0\alpha} \,\|.$$

Since $O(|\, T_{\alpha}{}^{n} \,|;\, T_{\alpha}{}^{p})$ is convex, it also contains the closed convex hulls of all the skeletons of the prisms mentioned above. The union of the closed convex hulls of the skeletons of the prisms spanned by all the simplexes $T_{\beta1}^q$ satisfying the conditions of 5.31 is a compactum (even a polyhedron) $\Pi$ contained in $G_\alpha = \| K_\alpha \| \setminus \| K_{0\alpha} \|$, and $\Pi$ contains the prism spanned by the cycles $z_{\beta1}^q$ and $S_{\alpha1}^{\beta1} z_{\beta1}^q$. Therefore,

$$z_{\beta1}^q \sim S_{\alpha1}^{\beta1} z_{\beta1}^q \quad \text{on } \Pi$$

(in the sense of XI, 4.6), i.e.,

$$s_{\beta1}^{\beta} D_\beta^q \bar{S}_\beta^\alpha z_\alpha^p \sim S_{\alpha1}^{\beta1} s_{\beta1}^{\beta} D_\beta^q \bar{S}_\beta^\alpha z_\alpha^p \quad \text{on } \Pi, \tag{5.32}$$

because of the definition of $z_{\beta1}^q$ in 5.31. By (5.2), we have

$$S_{\alpha1}^{\beta1} s_{\beta1}^{\beta} D_\beta^q \bar{S}_\beta^\alpha z_\alpha^p = s_{\alpha1}^{\alpha} D_\alpha^q z_\alpha^p.$$

Furthermore, since $s_{\alpha1}^{\alpha} D_\alpha^q z_\alpha^p$ is contained in $\Pi$,

$$s_{\beta1}^{\alpha} D_\alpha^q z_\alpha^p = s_{\beta1}^{\alpha1} s_{\alpha1}^{\alpha} D_\alpha^q z_\alpha^p \sim s_{\alpha1}^{\alpha} D_\alpha^q z_\alpha^p \quad \text{on } \Pi,$$

so that $S_{\alpha1}^{\beta1} s_{\beta1}^{\beta} D_\beta^q \bar{S}_\beta^\alpha z_\alpha^p$ can be replaced by $s_{\beta1}^{\alpha} D_\alpha^q z_\alpha^p$ in (5.32). We then obtain

$$s_{\beta1}^{\beta} D_\beta^q \bar{S}_\beta^\alpha z_\alpha^p - s_{\beta1}^{\alpha} D_\alpha^q z_\alpha^p \sim 0 \quad \text{on } \Pi.$$

As we saw in 4.2, there is a continuous mapping $C$ of $G_\beta$ onto $\| Q_{\beta1} \|$ which leaves all the points of $\| Q_{\beta1} \|$ fixed, and which consequently maps the cycle $s_{\beta1}^{\beta} D_\beta^q \bar{S}_\beta^\alpha z_\alpha^p - s_{\beta1}^{\alpha} D_\alpha^q z_\alpha^p$ on $Q_{\beta1}$ into itself. Therefore,

$$s_{\beta1}^{\beta} D_\beta^q \bar{S}_\beta^\alpha z_\alpha^p - s_{\beta1}^{\alpha} D_\alpha^q z_\alpha^p \sim 0 \quad \text{in } C(\Pi) \subset \| Q_{\beta1} \|. \tag{5.33}$$

But

$$s_{\beta1}^{\beta} D_\beta^q \bar{S}_\beta^\alpha z_\alpha^p - s_{\beta1}^{\alpha} D_\alpha^q z_\alpha^p \in Z_\Delta^q(Q_{\beta1}),$$

so that (5.33) implies (by XI, Corollary 4.61) that

$$s_{\beta1}^{\beta} D_\beta^q \bar{S}_\beta^\alpha z_\alpha^p - s_{\beta1}^{\alpha} D_\alpha^q z_\alpha^p \sim 0 \quad \text{in } Q_{\beta1}.$$

This completes the proof.

## §6. The spectrum $\mathfrak{K}_0$ of the set $\Phi$

**§6.1. The coverings $\Omega_\alpha$ and $\omega_\alpha$.** Retaining the notation of 4.1, we denote by $\Omega_\alpha$ the open covering of $S^n$ consisting of the open stars (IV, 3.2) of $K_\alpha$:

$$\Omega_\alpha = \{O_1^\alpha, \cdots, O_{s_\alpha}^\alpha\}.$$

We know that $K_\alpha$ is the nerve of the covering $\Omega_\alpha$. Suppose that $O_1^\alpha, \cdots, O_{s_{0\alpha}}^\alpha$ are those elements of $\Omega_\alpha$ which intersect $\Phi$. We set

$$o_i^\alpha = \Phi \cap O_i^\alpha \qquad (1 \leq i \leq s_{0\alpha}). \tag{6.11}$$

The open sets $o_1^\alpha, \cdots, o_{s_{0\alpha}}^\alpha$ of $\Phi$ form a covering of $\Phi$ which we denote by $\omega_\alpha$.

6.12. *The complex $K_{0\alpha}$ is the nerve of the covering $\omega_\alpha$.*

For, the vertices of $K_{0\alpha}$ are precisely those vertices of $K_\alpha$ whose open stars (relative to $K_\alpha$) intersect $\Phi$. In other words, the vertices of $K_{0\alpha}$ correspond (1—1) to the elements of $\omega_\alpha$ (a vertex $e$ of $K_{0\alpha}$ corresponds to the element $\Phi \cap O(K_\alpha\, ;\, e)$ of $\omega_\alpha$ [where $O(K_\alpha\, ;\, e)$ is the star of $e$ in $K_\alpha$]. [The elements of $\omega_\alpha$ form an indexed set (I, 1.3); some of them may coincide geometrically.]

Moreover, the intersection of the open stars of a set of vertices ${e_{i_0}}^\alpha, \cdots, {e_{i_r}}^\alpha$ of $K_\alpha$ is nonempty if, and only if, there is a simplex $T_\alpha$ with vertices ${e_{i_0}}^\alpha, \cdots, {e_{i_r}}^\alpha$ in $K_\alpha$; and then

$$O(K_\alpha\, ;\, {e_{i_0}}^\alpha) \cap \cdots \cap O(K_\alpha\, ;\, {e_{i_r}}^\alpha) = O(K_\alpha\, ;\, T_\alpha).$$

Therefore, the nerve of $\omega_\alpha$ consists of precisely those simplexes of $K_\alpha$ whose stars intersect $\Phi$, i.e., those which are faces of simplexes of $K_\alpha$ whose intersection with $\Phi$ is nonempty. But then $K_{0\alpha}$ is the nerve of $\omega_\alpha$.

It is clear that if $\beta > \alpha$, then $\Omega_\beta$ is a refinement of $\Omega_\alpha$ and consequently $\omega_\beta$ is a refinement of $\omega_\alpha$.

On the other hand, Condition 2° of 4.1 implies that there is a simplex $T_{0\alpha} \in K_{0\alpha}$ such that the diameter of $\Phi \cap T_{0\alpha}$ is greater than twice the diameter of an arbitrary $T_\beta \in K_\beta$.

Hence, there is an element of $\omega_\alpha$ which is not contained in any element of $\Omega_\beta$ and therefore in no one single element of $\omega_\beta$. Hence $\omega_\alpha$ is not a refinement of $\omega_\beta$.

We have proved

6.13. *If $\beta > \alpha$, $\omega_\beta > \omega_\alpha$.*

§**6.2. The spectrum $\mathcal{K}_0$ of the compactum $\Phi$.** Let us assign to each element ${O_j}^{\alpha+1}$ of the covering $\Omega_{\alpha+1}$ a definite element ${O_i}^\alpha = {S_\alpha}^{\alpha+1}({O_j}^{\alpha+1})$ of $\Omega_\alpha$ which contains ${O_j}^{\alpha+1}$.

This assignment defines a projection of $\Omega_{\alpha+1}$ into $\Omega_\alpha$ and also a projection ${S_{0\alpha}}^{0,\alpha+1}$ of $\omega_{\alpha+1}$ into $\omega_\alpha$: the element ${o_j}^{\alpha+1} = \Phi \cap {O_j}^{\alpha+1}$ of $\omega_{\alpha+1}$ is assigned the element ${o_i}^\alpha = {S_{0\alpha}}^{0,\alpha+1}{o_j}^{\alpha+1} = \Phi \cap {S_\alpha}^{\alpha+1}{O_j}^{\alpha+1}$ of $\omega_\alpha$; since ${O_i}^\alpha \supseteq {O_j}^{\alpha+1}$, ${o_i}^\alpha \supseteq {o_j}^{\alpha+1}$.

For $\beta > \alpha + 1$, we set

$${S_\alpha}^\beta = {S_\alpha}^{\alpha+1} \cdots {S_{\beta-2}}^{\beta-1}{S_{\beta-1}}^\beta; \qquad {S_{0\alpha}}^{0\beta} = {S_{0\alpha}}^{0,\alpha+1} \cdots {S_{0,\beta-2}}^{0,\beta-1}{S_{0,\beta-1}}^{0\beta}.$$

We have therefore defined for every pair of natural numbers $\alpha$ and $\beta$, $\beta > \alpha$, projections ${S_\alpha}^\beta$ of $\Omega_\beta$ into $\Omega_\alpha$ and ${S_{0\alpha}}^{0\beta}$ of $\omega_\beta$ into $\omega_\alpha$, such that

$$(6.21) \qquad {S_\alpha}^\gamma = {S_\alpha}^\beta {S_\beta}^\gamma, \qquad {S_{0\alpha}}^{0\gamma} = {S_{0\alpha}}^{0\beta}{S_{0\beta}}^{0\gamma}$$

for $\gamma > \beta > \alpha$.

The projections of the coverings induce simplicial mappings (denoted by the same letters) of the nerves of the coverings, i.e., a projection ${S_\alpha}^\beta$ of $K_\beta$ into $K_\alpha$ and a projection ${S_{0\alpha}}^{0\beta}$ of $K_{0\beta}$ into $K_{0\alpha}$, with ${S_\alpha}^\beta$ and ${S_{0\alpha}}^{0\beta}$ coinciding on $K_{0\beta}$.

The identity (6.21) shows that these projections satisfy Condition 2° of 1.1; Condition 1° of 1.1 is also satisfied, since the simplicial mappings $S_\alpha{}^\beta$, $S_{0\alpha}{}^{0\beta}$ are induced by the corresponding projections of the coverings $\Omega_\beta$, $\omega_\beta$ into $\Omega_\alpha$, $\omega_\alpha$. Hence, *the sequence of complexes*

$$\{K_{0\alpha}\} \qquad\qquad (\alpha = 1, 2, \cdots)$$

*together with the projections* $S_{0\alpha}{}^{0\beta}$ *just now defined on them form a spectrum of* $\Phi$. We denote this spectrum by $\mathfrak{K}_0$. Since $\nabla^p(\Phi) = \nabla^p(\mathfrak{K}_0)$, *to prove the Alexander-Pontryagin duality it is sufficient to show that* $\nabla^p(\mathfrak{K}_0) \approx \Delta^{q-1}(\Gamma)$. We shall do so in §8.

**§6.3. The projections $S_\alpha{}^\beta$.** We show, finally, that the projection $S_\alpha{}^\beta$ of $K_\beta$ into $K_\alpha$ is a normal simplicial mapping, i.e., that it has the following property:

6.31. *If $e_\beta$ is an arbitrary vertex of $K_\beta$, $\beta > \alpha$, then $S_\alpha{}^\beta e_\beta$ is a vertex of the carrier of $e_\beta$ in $K_\alpha$.*

For, according to the definition of $S_\alpha{}^\beta$,

$$O(K_\beta\,;\,e_\beta) \subset O(K_\alpha\,;\,S_\alpha{}^\beta e_\beta),$$

$$e_\beta \in \|\,T_\alpha\,\| \subset O(K_\alpha\,;\,S_\alpha{}^\beta e_\beta),$$

so that $S_\alpha{}^\beta e_\beta$ is a vertex of the carrier $T_\alpha \in K_\alpha$ of $e_\beta$.

COROLLARY 6.32. For arbitrary $T_\beta \in K_\beta$, the simplex $S_\alpha{}^\beta T_\beta$ is a face of the carrier of $T_\beta$ in $K_\alpha$.

In fact, if $T_\beta \subset T_\alpha \in K_\alpha$, then the carrier of each vertex of $T_\beta$ is a face of $T_\alpha$; hence, by 6.31, the image of each vertex of $T_\beta$ is a vertex of $T_\alpha$, which proves 6.32.

## §7. Further auxiliary propositions

**§7.1. Commutativity of the operators $\bar{S}_\beta{}^\alpha$ and $\nabla E$** (see VII, 5.6 for the definition of $E$). Let us consider the complexes $K_\alpha$, $K_\beta$ and their subcomplexes $K_{0\alpha}$ and $K_{0\beta}$. The projection $S_\alpha{}^\beta$ of $K_\beta$ into $K_\alpha$ induces a homomorphism $\bar{S}_\beta{}^\alpha$ of $L^p(K_\alpha)$ into $L^p(K_\beta)$, and the same projection restricted to the complex $K_{0\alpha} \subset K_\alpha$ induces a homomorphism $\bar{S}_{0\beta}{}^{0\alpha}$ of $L^p(K_{0\alpha})$ into $L^p(K_{0\beta})$. Then

7.1. *The relation*

$$\bar{S}_\beta{}^\alpha \nabla E z_{0\alpha}{}^p \backsim \nabla E \bar{S}_{0\beta}{}^{0\alpha} z_{0\alpha}{}^p \quad \text{in } K_\alpha \setminus K_{0\alpha} \tag{7.1}$$

*holds for every $z_{0\alpha}{}^p \in Z_\nabla{}^p(K_{0\alpha})$.*

*Proof.* Since the operators $\bar{S}_\beta{}^\alpha$ and $\nabla$ commute [see IX, (6.2)], we may rewrite (7.1) as

$$\nabla \bar{S}_\beta{}^\alpha E z_{0\alpha}{}^p \backsim \nabla E \bar{S}_{0\beta}{}^{0\alpha} z_{0\alpha}{}^p \quad \text{in } K_\alpha \setminus K_{0\alpha},$$

or in the form

$$\nabla(\bar{S}_\beta{}^\alpha E z_{0\alpha}{}^p - E \bar{S}_{0\beta}{}^{0\alpha} z_{0\alpha}{}^p) \backsim 0 \quad \text{in } K_\alpha \setminus K_{0\alpha}. \tag{7.11}$$

But (7.11) obviously follows from

$$(7.12) \qquad \bar{S}_\beta{}^\alpha E z_{0\alpha}{}^p - E\bar{S}_{0\beta}{}^{0\alpha} z_{0\alpha}{}^p \in L^p(K_\alpha \setminus K_{0\alpha}).$$

We shall prove (7.12).

For arbitrary $T_\beta{}^p \in K_\beta$ and arbitrary orientation $t_\beta{}^p$ of $T_\beta{}^p$,

$$(\bar{S}_\beta{}^\alpha E z_{0\alpha}{}^p \cdot t_\beta{}^p) = (E z_{0\alpha}{}^p \cdot S_\alpha{}^\beta t_\beta{}^p);$$

hence if $S_\alpha{}^\beta T_\beta{}^p \in K_{0\alpha}$, then

$$(\bar{S}_\beta{}^\alpha E z_{0\alpha}{}^p \cdot t_\beta{}^p) = (z_{0\alpha}{}^p \cdot S_\alpha{}^\beta t_\beta{}^p).$$

On the other hand, if $T_\beta{}^p \in K_{0\beta}$, then

$$(E\bar{S}_{0\beta}{}^{0\alpha} z_{0\alpha}{}^p \cdot t_\beta{}^p) = (\bar{S}_{0\beta}{}^{0\alpha} z_{0\alpha}{}^p \cdot t_\beta{}^p) = (z_{0\alpha}{}^p \cdot S_\alpha{}^\beta t_\beta{}^p).$$

But if $T_\beta{}^p \in K_{0\beta}$, then $S_\alpha{}^\beta T_\beta{}^p \in K_{0\alpha}$; hence,

$$(\bar{S}_\beta{}^\alpha E z_{0\alpha}{}^p \cdot t_\beta{}^p) = (E\bar{S}_{0\beta}{}^{0\alpha} z_{0\alpha}{}^p \cdot t_\beta{}^p)$$

for $T_\beta{}^p \in K_{0\beta}$, and (7.12) follows.

**§7.2. Corollary to (5.1) and (7.1).** The following proposition, a simple consequence of (5.1) and (7.1), will have great significance in the proof of the Alexander-Pontryagin duality.

7.2. *If* $z_{0\alpha}{}^p \in Z_\nabla{}^p(K_{0\alpha})$, *then*

$$(7.2) \qquad s_{\beta 1}{}^\beta \Delta D_\beta{}^q \bar{S}_{0\beta}{}^{0\alpha} z_{0\alpha}{}^p \sim s_{\beta 1}{}^\alpha \Delta D_\alpha{}^q z_{0\alpha}{}^p \quad \textit{in } Q_{\beta 1}.$$

The significance of (7.2) lies in the fact that it is used to prove that the homomorphisms $\Delta D_\alpha{}^q$ of the groups $Z_\nabla{}^p(K_{0\alpha})$ induce a mapping of $\nabla^p(\Phi)$ onto $\Delta^{q-1}(\Gamma)$ (see the proof of Lemma 8.1 and the paragraph following it).

*Proof of* (7.2). Substituting

$$z_\alpha{}^{p+1} = \nabla E z_{0\alpha}{}^p$$

into (5.1), we obtain

$$(7.21) \qquad s_{\beta 1}{}^\beta D_\beta{}^{q-1} \bar{S}_\beta{}^\alpha \nabla E z_{0\alpha}{}^p \sim s_{\beta 1}{}^\alpha D_\alpha{}^{q-1} \nabla E z_{0\alpha}{}^p \quad \text{in } Q_{\beta 1}.$$

But, in view of (7.1),

$$\bar{S}_\beta{}^\alpha \nabla E z_{0\alpha}{}^p \backsim \nabla E \bar{S}_{0\beta}{}^{0\alpha} z_{0\alpha}{}^p \quad \text{in } K_\alpha \setminus K_{0\alpha},$$

so that (see XIII, $3.322_\nabla$)

$$D_\beta{}^{q-1} \bar{S}_\beta{}^\alpha \nabla E z_{0\alpha}{}^p \sim D_\beta{}^{q-1} \nabla E \bar{S}_{0\beta}{}^{0\alpha} z_{0\alpha}{}^p \quad \text{in } K_\alpha{}^* \setminus K_{0\alpha}{}^*;$$

hence $D_\beta{}^{q-1} \bar{S}_\beta{}^\alpha \nabla E z_{0\alpha}{}^p$ can be replaced by $D_\beta{}^{q-1} \nabla E \bar{S}_{0\beta}{}^{0\alpha} z_{0\alpha}{}^p$ on the left side of (7.21). This yields

$$(7.22) \qquad s_{\beta 1}{}^\beta D_\beta{}^{q-1} \nabla E \bar{S}_{0\beta}{}^{0\alpha} z_{0\alpha}{}^p \sim s_{\beta 1}{}^\alpha D_\alpha{}^{q-1} \nabla E z_{0\alpha}{}^p \quad \text{in } Q_{\beta 1}.$$

Now XIII, (3.32) implies that

$$D_\beta{}^{q-1}\nabla E\bar{S}_{0\beta}{}^{0\alpha}z_{0\alpha}{}^p = (-1)^{p+1}\Delta D_\beta{}^q E\bar{S}_{0\beta}{}^{0\alpha}z_{0\alpha}{}^p = (-1)^{p+1}\Delta D_\beta{}^q\bar{S}_{0\beta}{}^{0\alpha}z_{0\alpha}{}^p,$$

$$D_\alpha{}^{q-1}\nabla Ez_{0\alpha}{}^p = (-1)^{p+1}\Delta D_\alpha{}^q Ez_{0\alpha}{}^p = (-1)^{p+1}\Delta D_\alpha{}^q z_{0\alpha}{}^p.$$

Inserting these expressions in (7.22) and dividing both sides by $(-1)^{p+1}$, we obtain

$$s_{\beta 1}{}^{\beta}\Delta D_\beta{}^q\bar{S}_{0\beta}{}^{0\alpha}z_{0\alpha}{}^p \sim s_{\beta 1}{}^{\alpha}\Delta D_\alpha{}^q z_{0\alpha}{}^p \quad \text{in } Q_{\beta 1}.$$

This is what we were to prove.

## §8. Proof of the Alexander-Pontryagin duality

**§8.1. The homomorphism $\Delta D^q$ of $\nabla^p(\Phi)$ into $\Delta^{q-1}(\Gamma)$ induced by the homomorphisms $\Delta D_\alpha{}^q$ of the groups $Z_\Delta{}^p(K_{0\alpha})$ into the groups $\Delta^{q-1}(K_\alpha{}^* \setminus K_{0\alpha}{}^*)$.** Let us assign to each cocycle $z_{0\alpha}{}^p \in Z_\nabla{}^p(K_{0\alpha})$ the chain $\Delta D_\alpha{}^q z_{0\alpha}{}^p$. We know XIII, (4.41)] that

$$\Delta D_\alpha{}^q z_{0\alpha}{}^p \in Z_\Delta{}^{q-1}(K_\alpha{}^* \setminus K_{0\alpha}{}^*),$$

and that consequently,

$$s_{\alpha 1}{}^{\alpha}\Delta D_\alpha{}^q z_{0\alpha}{}^p \in Z_\Delta{}^{q-1}(\Gamma). \tag{8.11}$$

LEMMA 8.1. *If two cocycles $z_{0\alpha}{}^p \in Z_\nabla{}^p(K_{0\alpha})$ and $z_{0\beta}{}^p \in Z_\nabla{}^p(K_{0\beta})$ belong to the same cohomology class of the spectrum $\mathfrak{K}_0$ of $\Phi$, then*

$$s_{\alpha 1}{}^{\alpha}\Delta D_\alpha{}^q z_{0\alpha}{}^p \sim s_{\beta 1}{}^{\beta}\Delta D_\beta{}^q z_{0\beta}{}^p \quad \textit{in } \Gamma.$$

*Proof*. Since $z_{0\alpha}{}^p$ and $z_{0\beta}{}^p$ are in the same cohomology class of the spectrum $\mathfrak{K}_0$, there is a $\gamma$, $\gamma > \alpha$, $\gamma > \beta$, such that [see XIII, (4.211)]

$$\bar{S}_{0\gamma}{}^{0\alpha}z_{0\alpha}{}^p - \bar{S}_{0\gamma}{}^{0\beta}z_{0\beta}{}^p \in C'_{K_\gamma}{}^p(K_{0\gamma}). \tag{8.12}$$

To prove the lemma, it is sufficient to show that

$$s_{\gamma 1}{}^{\alpha}\Delta D_\alpha{}^q z_{0\alpha}{}^p \sim s_{\gamma 1}{}^{\beta}\Delta D_\beta{}^q z_{0\beta}{}^p \quad \text{in } \Gamma. \tag{8.13}$$

We shall now prove (8.13).

By (8.12) and XIII, 4.420,

$$\Delta D_\gamma{}^q\bar{S}_{0\gamma}{}^{0\alpha}z_{0\alpha}{}^p - \Delta D_\gamma{}^q\bar{S}_{0\gamma}{}^{0\beta}z_{0\beta}{}^p \in H_\Delta{}^{q-1}(K_\gamma{}^* \setminus K_{0\gamma}{}^*),$$

and, therefore,

$$s_{\gamma 1}{}^{\gamma}\Delta D_\gamma{}^q\bar{S}_{0\gamma}{}^{0\alpha}z_{0\alpha}{}^p \sim s_{\gamma 1}{}^{\gamma}\Delta D_\gamma{}^q\bar{S}_{0\gamma}{}^{0\beta}z_{0\beta}{}^p \quad \text{in } \Gamma. \tag{8.14}$$

But (7.2) yields (with $\beta$ replaced by $\gamma$)

$$s_{\gamma 1}{}^{\gamma}\Delta D_\gamma{}^q\bar{S}_{0\gamma}{}^{0\alpha}z_{0\alpha}{}^p \sim s_{\gamma 1}{}^{\alpha}\Delta D_\alpha{}^q z_{0\alpha}{}^p \quad \text{in } \| Q_{\gamma 1} \| \subset \Gamma.$$

Replacing $\alpha$ by $\beta$ and $\beta$ by $\gamma$ in (7.2), we obtain

$$s_{\gamma 1}{}^{\gamma}\Delta D_\gamma{}^q\bar{S}_{0\gamma}{}^{0\beta}z_{0\beta}{}^p \sim s_{\gamma 1}{}^{\beta}\Delta D_\beta{}^q z_{0\beta}{}^p \quad \text{in } \| Q_{\gamma 1} \| \subset \Gamma,$$

so that (8.14) becomes

$$s_{\gamma 1}{}^{\alpha}\Delta D_{\alpha}{}^{q} z_{0\alpha}{}^{p} \sim s_{\gamma 1}{}^{\beta}\Delta D_{\beta}{}^{q} z_{0\beta}{}^{p} \quad \text{in } \Gamma.$$

This proves Lemma 8.1.

Now let $\zeta^p$ be an arbitrary cohomology class of the spectrum $\mathcal{K}_0$. Choose an arbitrary $z_{0\alpha}{}^{p} \in Z_{\nabla}{}^{p}(K_{0\alpha})$ in $\zeta^p$, and denote by $\Delta D^q \zeta^p$ the homology class $\mathfrak{z}^{q-1} \in \Delta^{q-1}(\Gamma)$ which contains the cycle $s_{\alpha 1}{}^{\alpha}\Delta D_{\alpha}{}^{q} z_{0\alpha}{}^{p}$. It follows from Lemma 8.1 that $\mathfrak{z}^{q-1}$ is independent of the choice of $z_{0\alpha}{}^{p} \in \zeta^p$; hence $\Delta D^q$ is a homomorphism of $\nabla^p(\mathcal{K}_0)$ [i.e., of the group $\nabla^p(\Phi)$] into $\Delta^{q-1}(\Gamma)$.

**§8.2. The mapping $\Delta D^q$ is an isomorphism.** Suppose that

$$\zeta^p \in \nabla^p(\mathcal{K}_0), \qquad \Delta D^q \zeta^p = 0.$$

We must prove that

$$\zeta^p = 0.$$

Since

$$\Delta D^q \zeta^p = 0$$

by hypothesis, there is a $z_{0\alpha}{}^{p} \in \zeta^p$ such that

$$s_{\alpha 1}{}^{\alpha}\Delta D_{\alpha}{}^{q} z_{0\alpha}{}^{p} \sim 0 \quad \text{in } \Gamma;$$

therefore,

$$s_{\beta 1}{}^{\alpha}\Delta D_{\alpha}{}^{q} z_{0\alpha}{}^{p} \sim 0 \quad \text{in } \| Q_{\beta 1} \|$$

for sufficiently large $\beta \geq \alpha$, and, consequently, $s_{\beta 1}{}^{\alpha}\Delta D_{\alpha}{}^{q} z_{0\alpha}{}^{p} \sim 0$ in $Q_{\beta 1}$ (because of XI, Theorem 4.6).

By (7.2),

$$s_{\beta 1}{}^{\alpha}\Delta D_{\alpha}{}^{q} z_{0\alpha}{}^{p} \sim s_{\beta 1}{}^{\beta}\Delta D_{\beta}{}^{q} \bar{S}_{0\beta}{}^{0\alpha} z_{0\alpha}{}^{p} \quad \text{in } Q_{\beta 1},$$

so that

$$s_{\beta 1}{}^{\beta}\Delta D_{\beta}{}^{q} \bar{S}_{0\beta}{}^{0\alpha} z_{0\alpha}{}^{p} \sim 0 \quad \text{in } Q_{\beta 1}.$$

Consequently, by 4.32,

$$\Delta D_{\beta}{}^{q} \bar{S}_{0\beta}{}^{0\alpha} z_{0\alpha} \sim 0 \quad \text{in } K_{\beta}{}^{*} \qquad K_{0\beta}{}^{*}.$$

It follows from this and XIII, 4.430 that

$$\bar{S}_{0\beta}{}^{0\alpha} z_{0\alpha}{}^{p} \in C_{K_\beta}{}^{p}(K_{0\beta}). \tag{8.2}$$

But $\bar{S}_{0\beta}{}^{0\alpha} z_{0\alpha}{}^{p} \in \zeta^p$ and $C_{K_\beta}{}^{p}(K_{0\beta}) = H_{\nabla}{}^{p}(K_{0\beta})$ for $p \geq 1$ [XIII, (4.21)]. Therefore, in view of (8.2), $\zeta^p = 0$, which completes the proof.

**§8.3. The isomorphism $\Delta D^q$ is onto.** Suppose that

$$\mathfrak{u}^{q-1} \in \Delta^{q-1}(\Gamma).$$

In view of 4.31, there is a cycle $u_{\alpha 1}{}^{q-1} \in \mathfrak{u}^{q-1}$ of the form

$$u_{\alpha 1}{}^{q-1} = s_{\alpha 1}{}^{\alpha} u_{\alpha}{}^{q-1},$$

where

$$u_\alpha{}^{q-1} \in Z_\Delta{}^{q-1}(K_\alpha{}^* \setminus K_{0\alpha}{}^*).$$

According to XIII, 4.440 and (4.22), there is a cocycle $z_{0\alpha}{}^p \in Z_\nabla{}^p(K_{0\alpha})$ such that

$$\Delta D_\alpha{}^q z_{0\alpha}{}^p \sim u_\alpha{}^{q-1} \quad \text{in } K_\alpha{}^* \setminus K_{0\alpha}{}^*;$$

consequently,

$$s_{\alpha 1}{}^\alpha \Delta D_\alpha{}^q z_{0\alpha}{}^p \in \mathfrak{u}^{q-1}.$$

If $\zeta^p \in \nabla(\mathfrak{K}_0)$ is the cohomology class of the spectrum $\mathfrak{K}_0$ which contains $z_{0\alpha}{}^p$, then

$$\Delta D^q \zeta^p = \mathfrak{u}^{q-1}.$$

This completes the proof of the Alexander-Pontryagin duality.

## Chapter XV

# LINKING. THE LITTLE ALEXANDER DUALITY

**§0.1. Introductory remarks.** The coefficient domains in this chapter are rings with unity.

All the constructions of this chapter and many of the next chapter are based on the complexes $K(R^n)$ and $K(S^n)$, i.e., the complex $K(\Gamma)$ with $\Gamma = R^n$ or $S^n$ (see IV, 1.5, Example 2 or XIV, 3.1).

In this and the next chapter $\Phi$ denotes a closed bounded subset of $R^n$, and *we shall always regard an $\epsilon$-chain of the compactum $\Phi$ as a chain of $R^n$.*

In other words, if $x^r = \sum a_i t_i^r$ is an $\epsilon$-chain of $\Phi \subset R^n$, we take $t_i^r$ to mean an oriented (perhaps degenerate) simplex of the complex $K(R^n)$, defined by the corresponding skeleton of $\Phi \subset R^n$. This refers, in particular, to the elements $z_k^r$ of the proper cycles $\mathfrak{z}^r = (z_1^r, z_2^r, \cdots, z_k^r, \cdots)$ of $\Phi \subset R^n$ (see XI). The important notion of the body $\bar{x}^r$ of a chain $x^r$ was defined in VII, 5.2, Remark 3 (and repeated in XIV, 3.1, Remark 2).

We say that two chains $x$ and $y$ are *in relative general position* (*with respect to each other*) if at least one of the following two conditions holds for every simplex $T_1$ in $x$ and every simplex $T_2$ in $y$: a) the closed convex hulls of the skeletons of $T_1$ and $T_2$ do not intersect, b) the simplexes $T_1$ and $T_2$ are nondegenerate and their vertices form a system of points in general position (see Appendix 1, 1.4).

For arbitrary natural number $n$, we use $p$ and $q$ to denote nonnegative integers whose sum is $n$.

## §1. The intersection number and the linking number

**§1.1. The intersection number of two chains $x^p$ and $y^q$ of $R^n$.** Let

$$x^p = \sum a_i t_i^p, \qquad y^q = \sum b_j t_j^q$$

be two chains of $R^n$ in relative general position. Their *intersection number* $(x^p \times y^q)$ is, by definition,

$$(x^p \times y^q) = \sum\nolimits_{i,j} (t_i^p \times t_j^q) a_i b_j , \tag{1.1}$$

where $(t_i^p \times t_j^q)$ is the intersection number (equal to $+1$, $-1$ or $0$) defined in VII, 2.2 for pairs of oriented simplexes $t_i^p$ and $t_j^q$. Hence the intersection number of two chains is an element of the prescribed coefficient ring $\mathfrak{A}$.

The following proposition is very often useful:

1.11. *If $x^p$ and $y^q$ are two chains of $R^n$ in relative general position, there exists an $\epsilon > 0$ such that every $\epsilon$-displacement maps $x^p$ and $y^q$ into chains $x'^p$, $y'^q$ which are also in relative general position and such that*

$$(x'^p \times y'^q) = (x^p \times y^q).$$

For, let $T_k$ and $T_l$ be any two simplexes of the complex $|\,x^p\,| \cup |\,y^q\,|$ such that the distance $\rho_{kl}$ between the closed convex hulls of $T_k$ and $T_l$ is positive. Then to prove the assertion, it is enough to take $\epsilon$ less than one half the minimum of the $\rho_{kl}$.

**§1.2. Computation formulas for intersection numbers.** It follows from VII, Theorem 2.1 that

$$(x^p \times y^q) = (-1)^{pq}(y^q \times x^p). \tag{1.21}$$

The following formulas follow immediately from Def. (1.1):

$$\begin{aligned}(x^p \times [y_1^q + y_2^q]) &= (x^p \times y_1^q) + (x^p \times y_2^q),\\ ([x_1^p + x_2^p] \times y^q) &= (x_1^p \times y^q) + (x_2^p \times y^q);\end{aligned} \tag{1.22}$$

$$(ax^p \times y^q) = (x^p \times ay^q) = a(x^p \times y^q). \tag{1.23}$$

The following proposition is very important:

THEOREM 1.2. *If $n_1 = r$, $n_2 = s$, $r + s = n + 1$ and the chains $x^r$, $y^s$ of $R^n$ are in relative general position, then*

$$(x^r \times \Delta y^s) = (-1)^r(\Delta x^r \times y^s). \tag{1.24}$$

*Proof.* Because of (1.1), (1.22) and (1.23), to prove (1.24) it is enough to prove the analogous formula for two oriented simplexes:

$$(t_1^r \times \Delta t_2^s) = (-1)^r(\Delta t_1^r \times t_2^s). \tag{1.240}$$

If the closures of the simplexes $T_1^r = |\,t_1^r\,|$ and $T_2^s = |\,t_2^s\,|$ are disjoint, both sides of (1.240) are zero and the formula is valid.

Suppose that $\bar{T}_1^r \cap \bar{T}_2^s \neq 0$. Then the intersection of the carrying planes of the two simplexes is a straight line, because of the assumption that they are in relative general position; and the intersection of the two simplexes themselves is an (open) segment, since they are bounded convex sets, i.e.,

$$T_1^r \cap T_2^s = (o_1 o_2).$$

Each of the two points $o_1$, $o_2$ lies on the boundary of one of the simplexes $T_1^r$, $T_2^s$ and is an interior point of the other simplex. We distinguish between two essentially different cases:

1°. Both points $o_1$ and $o_2$ are on the boundary of a single one of the simplexes $T_1^r$, $T_2^s$.

2°. $o_1$ is on the boundary of $T_1^r$ and $o_2$ is on the boundary of $T_2^s$.

*Proof of* (1.240): *Case* 1°. (Fig. 150 shows the case $r = s = 2$.) Suppose that both $o_1$ and $o_2$ are on the boundary of $T_1^r$. This assumption is not restrictive, since, notwithstanding the apparent asymmetry of (1.24) with

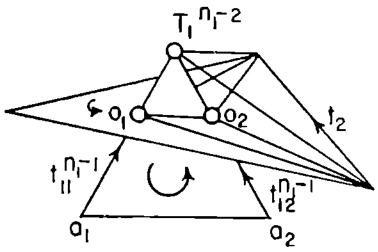


Fig. 150

respect to the chains $x^r$ and $y^s$, it is easy to verify that

$$(1.24') \qquad (y^s \times \Delta x^r) = (-1)^s(\Delta y^s \times x^r)$$

follows from (1.24) and (1.21).

Since the two simplexes are in relative general position, $o_1$ and $o_2$ are interior points of two distinct faces $T_{11}^{r-1}$ and $T_{12}^{r-1}$ of $T_1^r$. Let $T_1^{r-2}$ be the common face of $T_{11}^{r-1}$ and $T_{12}^{r-1}$; and let $a_1$, $a_2$ be vertices of $T_{11}^{r-1}$, $T_{12}^{r-1}$, respectively, which are not vertices of $T_1^{r-2}$. For arbitrary orientation $t_1^{r-2}$ of $T_1^{r-2}$, the simplexes

$$t_{11}^{r-1} = (a_1t_1^{r-2})$$

and

$$t_{12}^{r-1} = (a_2t_1^{r-2})$$

appear in $\Delta t_1^r$ with opposite signs (see VII, 3.2, Property 2). Since the left side of (1.240) is zero, because the points $o_1$ and $o_2$, in view of the relative general position of $t_1^r$ and $t_2^s$, must be interior points of $T_2^s$, (1.240) is equivalent to

$$(1.2401) \qquad (t_{11}^{r-1} \times t_2^s) = (t_{12}^{r-1} \times t_2^s).$$

To prove (1.2401) we choose an $(s - 1)$-simplex $|\,t_2\,|$ in the plane of $T_2^s$ such that the segment $(o_1o_2)$ does not intersect the plane of $|\,t_2\,|$ and such that the vertices of $|\,t_2\,|$ are in general position in $R^n$ with respect to the vertices of $T_1^{r-2}$ (in Fig. 150 the simplex $|\,t_2\,|$ is a face of $T_2^s$). Then

$$(t_{11}^{r-1} \times t_2^s) = [t_{11}^{r-1}(o_1t_1^{r-2})] \cdot [t_2^s(o_1t_2)] \cdot [R^n(o_1t_1^{r-2}t_2)],$$

$$(t_{12}^{r-1} \times t_2^s) = [t_{12}^{r-1}(o_2t_1^{r-2})] \cdot [t_2^s(o_2t_2)] \cdot [R^n(o_2t_1^{r-2}t_2)].$$

(For definitions and notation, see VII, 1.4, 1.5 and 2.1.) But, by VII, Theorem 1.51,

$$t_{11}^{r-1}(o_1t_1^{r-2}) = (a_1t_1^{r-2})(o_1t_1^{r-2}) = +1,$$

since the points $a_1$ and $o_1$ are on the same side of the plane determined by $t_1^{r-2}$ in the plane of $t_{11}^{r-1}$.

Analogously,

$$t_{12}^{r-1}(o_2t_1^{r-2}) = (a_2t_1^{r-2})(o_2t_1^{r-2}) = +1.$$

Consequently,

$$\begin{aligned} (t_{11}^{r-1} \times t_2^s) &= t_2^s(o_1t_2) \cdot R^n(o_1t_1^{r-2}t_2), \\ (t_{12}^{r-1} \times t_2^s) &= t_2^s(o_2t_2) \cdot R^n(o_2t_1^{r-2}t_2). \end{aligned} \tag{1.241}$$

Furthermore,

$$t_2^s(o_1t_2) = t_2^s(o_2t_2), \tag{1.242}$$

since $o_1$ and $o_2$ are on the same side of the plane of $t_2$ in the plane of $t_2^s$ [the segment $(o_1o_2)$ does not intersect the plane of $t_2$]. Finally,

$$R^n(o_1t_1^{r-2}t_2) = R^n(o_2t_1^{r-2}t_2), \tag{1.243}$$

since $o_1$ and $o_2$ are on the same side of the plane of $(t_1^{r-2}t_2)$ in $R^n$ [the intersection of the plane of $(t_1^{r-2}t_2)$ with the plane of $t_2^s$ is the plane of $t_2$ and the segment $(o_1o_2)$ does not meet the latter plane].

Formulas (1.241), (1.242) and (1.243) imply (1.2401), so that (1.240) follows and hence (1.24) for Case 1°.

*Proof of* (1.240): *Case* 2°. (See Fig. 151 for the case $r = s = 2$.) Suppose that the points $o_1$ and $o_2$ are on the boundary of the simplexes $T_1^r$ and $T_2^s$ respectively, and that they are contained in the faces $T_1^{r-1}$ and $T_2^{s-1}$ ($o_1$ and $o_2$ cannot belong to faces of lesser dimension because $T_1^r$ and $T_2^s$ are in relative general position). Let $a_i$ be a vertex of $T_i^{n_i}$ which does not belong to $T_i^{n_i-1}$ ($i = 1, 2$; $n_1 = r$, $n_2 = s$). If $t_1^{r-1}$ and $t_2^{s-1}$ are orientations such that

$$t_1^r = (a_1t_1^{r-1}), \qquad t_2^s = (a_2t_2^{s-1}),$$

then (1.240) becomes

$$(t_1^r \times t_2^{s-1}) = (-1)^r(t_1^{r-1} \times t_2^s). \tag{1.2402}$$

To prove (1.2402), let $b_1$ and $b_2$ be vertices of $|\, t_1^{r-1} \,|$ and $|\, t_2^{s-1} \,|$, respectively, and let $|\, t_1^{r-2} \,|$ and $|\, t_2^{s-2} \,|$ be the faces of $|\, t_1^{r-1} \,|$ and $|\, t_2^{s-1} \,|$ opposite

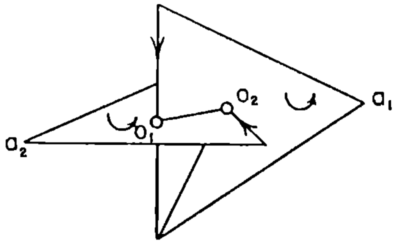


FIG. 151

the vertices $b_1$, $b_2$. Choose the orientations of $|\,t_1^{r-2}\,|$ and $|\,t_2^{s-2}\,|$ so that

$$t_1^{r-1} = (b_1t_1^{r-2}), \qquad t_2^{s-1} = (b_2t_2^{s-2}).$$

Since the segment $(o_ib_i)$ $(i = 1, 2)$ does not meet the plane of $|\,t_i^{n_i-2}\,|$, $o_i$ and $b_i$ are on the same side of the plane of the face $|\,t_i^{n_i-2}\,|$ in the plane of the face $|\,t_i^{n_i-1}\,|$; hence

$$(1.244) \qquad t_i^{n_i-1}(o_it_i^{n_i-2}) = +1 \qquad (i = 1, 2).$$

Furthermore, $o_2$ and $a_1$ are on the same side of the plane of the face $|\,t_1^{r-1}\,|$ in the plane of $|\,t_1^r\,|$ [the former is also the plane of the simplex $(o_1t_1^{r-2})$]; therefore, by VII, Theorem 1.5, the orientations of $(o_2o_1t_1^{r-2})$ and $t_1^r = (a_1t_1^{r-1})$ are coherent, i.e.,

$$(1.245) \qquad t_1^r(o_2o_1t_1^{r-2}) = +1.$$

Similarly,

$$(1.246) \qquad t_2^s(o_1o_2t_2^{s-2}) = t_2^s(o_1t_2^{s-1}) = +1.$$

We now compute

$$(t_1^r \times t_2^{s-1}) = [t_1^r(o_2o_1t_1^{r-2})] \cdot [t_2^{s-1}(o_2t_2^{s-2})] \cdot [R^n(o_2o_1t_1^{r-2}t_2^{s-2})],$$
$$(t_1^{r-1} \times t_2^s) = [t_1^{r-1}(o_1t_1^{r-2})] \cdot [t_2^s(o_1o_2t_2^{s-2})] \cdot [R^n(o_1t_1^{r-2}o_2t_2^{s-2})];$$

or, by (1.244)–(1.246),

$$(t_1^r \times t_2^{s-1}) = R^n(o_2o_1t_1^{r-2}t_2^{s-2}),$$
$$(t_1^{r-1} \times t_2^s) = R^n(o_1t_1^{r-2}o_2t_2^{s-2}).$$

But

$$(o_2o_1t_1^{r-2}t_2^{s-2})(o_1t_1^{r-2}o_2t_2^{s-2}) = (-1)^r,$$

from which (1.2402) follows.

This proves the theorem.

**§1.3. Corollaries of Theorem 1.2. Generalization of the definition of intersection number.**

THEOREM 1.31. *Suppose that a cycle $z^p$ and a chain $x^q$ of $R^n$ are in relative general position. If*

$$z^p \sim 0 \quad in\ R^n \setminus \overline{\Delta x}^q,$$

*then*

$$(z^p \times x^q) = 0.$$

(For the definition of the body $\bar{x}^r$ of a chain $x^r$ see VII, 5.2, Remark 3. For convenience of notation, we denote the body of $\Delta x^r$ by $\overline{\Delta x}^r$.)

*Proof.* Let $z^p = \Delta x^{p+1}$, where $x^{p+1}$ is in $R^n \setminus \bar{\Delta} x^q$. Let $\epsilon < \frac{1}{2}\rho(\bar{x}^{p+1}, \bar{\Delta} x^q)$ be a positive number chosen in accordance with 1.11 for the chains $z^p$ and $x^q$, and map the chains $x^{p+1}$ and $x^q$ into chains $x'^{p+1}$, $x'^q$ by means of an $\epsilon$-displacement so that the following condition is satisfied:

1°. The set of all vertices of both chains $x'^{p+1}$ and $x'^q$ are in general position.

Then, in view of 1.11, the following condition is also satisfied:

2°. If $z'^p = \Delta x'^{p+1}$, then

$$(z'^p \times x'^q) = (z^p \times x^q).$$

It is sufficient to prove that

$$(z'^p \times x'^q) = 0.$$

But $x'^{p+1}$ and $x'^q$ satisfy the conditions of Theorem 1.2; hence

$$(z'^p \times x'^q) = (\Delta x'^{p+1} \times x'^q) = \pm(x'^{p+1} \times \Delta x'^q) = 0$$

This is what we were to prove.

As a special case, we have

1.310. *If* $z_1^p$ *and* $z_2^q$ *are cycles in relative general position in* $R^n$, *then* $(z_1^p \times z_2^q) = 0$.

Indeed, suppose to begin with that $p < n$ (we also assume that if $p = 0$, $z_1^p$ is a normal cycle). Then $z_1^p \sim 0$ in $R^n$, $\Delta z_2^q = 0$ and 1.310 is a special case of 1.31.

Now let $p = 0$ (so that $q = n$), and suppose that $z_1^0$ is not a normal cycle. In view of (1.22) and (1.23), we may assume that $z_1^0$ is a point with coefficient $+1$, chosen so that it is in general position with respect to $\bar{z}_2^n$. If $y^0$ is a second point with coefficient $+1$, chosen in $R^n \setminus \bar{z}_2^n$ (the set $\bar{z}_2^n$ is bounded, so that $y^0$ exists), then

$$(y^0 \times z_2^n) = 0;$$

$z_1^0 - y^0$ is a normal cycle and we have proved above that

$$[(z_1^0 - y^0) \times z_2^n] = 0.$$

Hence, because of (1.22) and (1.23) it follows that

$$(z_1^0 \times z_2^n) = 0.$$

The remaining case $p = n$ does not require special proof, since in that case $q = 0$ and because the proposition is symmetric in $z_1^p$ and $z_2^q$, the theorem is also true in this case.

Theorem 1.2 leads to a significant generalization of the conditions in which the intersection number of two chains is defined: for $p + q = n$, the intersection number of two arbitrary chains $x^p$ and $x^q$ in $R^n$ is defined if

$$(1.32) \qquad \bar{x}^p \cap \overline{\Delta x}^q = 0, \qquad \overline{\Delta x}^p \cap \bar{x}^q = 0,$$

i.e., *if neither chain intersects the boundary of the other.*

To show this, we choose $\epsilon > 0$ so that $3\epsilon$ is less than $\rho(\bar{x}^p, \overline{\Delta x}^q), \rho(\overline{\Delta x}^p, \bar{x}^q)$ and the least distance between any pair of distinct vertices of $x^p$, $x^q$.

We subject the vertices of $x^p$ to an $\epsilon$-displacement $S'$ such that the chain $x'^p$ resulting from $x^p$ after this displacement is in relative general position with $x^q$.

The intersection number $(x'^p \times x^q)$ is defined and we set

$$(1.33_p) \qquad (x^p \times x^q) = (x'^p \times x^q).$$

We shall show that this definition is independent of the choice of the chain $x'^p$, i.e., of the $\epsilon$-displacement to which $x^p$ is subjected. Let $S''$ be a second $\epsilon$-displacement (satisfying the same conditions as $S'$) resulting in $x''^p$. We shall show that

$$(1.331) \qquad (x'^p \times x^q) = (x''^p \times x^q).$$

Since neither $S'$ nor $S''$ maps two distinct vertices of $x^p$ into a single vertex of $x'^p$ or $x''^p$, the chain $x''^p$ may be obtained from $x'^p$ by means of the $2\epsilon$-displacement $S = S''S'^{-1}$.

Enumerating the vertices of $x^p$ in a definite order and using the induced order of the vertices of $x'^p$ and $x''^p$, we consider the prism $\Pi x'^p$ spanned by $x'^p$ and its image $x''^p$ under $S$ (VII, 9.3). The entire prism is contained in a $3\epsilon$-neighborhood of the set $\bar{x}^p$, that is, in $R^n \setminus \overline{\Delta x}^q$. By VII, (9.34),

$$\Delta \Pi x'^p = x'^p - x''^p - \Pi \Delta x'^p;$$

hence

$$x'^p - x''^p - \Pi\Delta x'^p \sim 0 \quad \text{in } R^n \setminus \overline{\Delta x}^q,$$

and consequently, in view of Theorem 1.31,

$$([x'^p - x''^p - \Pi\Delta x'^p] \times x^q) = 0.$$

However, since $\Pi\Delta x'^p$ is contained in a $3\epsilon$-neighborhood of the set $\overline{\Delta x}^p$, i.e., in $R^n \setminus \bar{x}^q$, it follows that $(\Pi\Delta x'^p \times x^q) = 0$, i.e.,

$$([x'^p - x''^p] \times x^q) = 0, \qquad (x'^p \times x^q) = (x''^p \times x^q).$$

This completes the proof.

To supplement this result, we add the following remark. Let us map the chain $x^q$ into a chain $x'^q$ by means of an $\epsilon$-displacement so that $x'^q$ is in relative general position with $x^p$ ($\epsilon$ has its previous meaning). We shall prove that

$$(1.33_q) \qquad (x^p \times x^q) = (x^p \times x'^q),$$

where the left side is defined by $(1.33_p)$ and the right side is to be taken in the sense of 1.1. To prove $(1.33_q)$, we choose an $\epsilon$-displacement $x'^p$ of $x^p$ such that $x'^p$ is in relative general position with both $x^q$ and $x'^q$. Then the definition of the intersection numbers $(x'^p \times x^q)$, $(x'^p \times x'^q)$ (in the sense of 1.1) and arguments completely analogous to the preceding show that

$$(x'^p \times x'^q) = (x'^p \times x^q),$$
$$(x^p \times x'^q) = (x'^p \times x'^q),$$

so that

$$(x^p \times x'^q) = (x'^p \times x^q).$$

This is what we were to prove.

Hence,

1.33. *If neither one of the chains $x^p$, $x^q$ intersects the boundary of the other, the intersection number $(x^p \times x^q)$ can be defined by any of the formulas*

$$(1.33) \qquad \begin{aligned} &(x^p \times x^q) = (x'^p \times x^q), \qquad (x^p \times x^q) = (x^p \times x'^q), \\ &\qquad\qquad (x^p \times x^q) = (x'^p \times x'^q), \end{aligned}$$

*where $x'^p$ and $x'^q$ are arbitrary sufficiently small displacements of $x^p$ and $x^q$ which are in relative general position with all the chains appearing in* (1.33).

REMARK. Formulas (1.21)–(1.24) obviously remain valid for the generalized definition of intersection number.

**§1.4. The linking number.** Let $z_1^p$ and $z_2^{q-1}$ be two nonintersecting cycles in $R^n$, with $p + q = n$ and with $z_1^p$ a normal cycle if $p = 0$. Let $x^{p+1}$ be a chain such that

$$\Delta x^{p+1} = z_1^p.$$

Since (1.32) is satisfied, the intersection number $(x^{p+1} \times z_2^{q-1})$ is defined.

1.41. *The value of $(x^{p+1} \times z_2^{q-1})$ is independent of the choice of the chain $x^{p+1}$, i.e.,*

$$(1.41) \qquad (x_1^{p+1} \times z_2^{q-1}) = (x_1'^{p+1} \times z_2^{q-1})$$

*for two arbitrary chains $x_1^{p+1}$ and $x_1'^{p+1}$ such that $\Delta x_1^{p+1} = \Delta x_1'^{p+1} = z_1^p$.*

For, $x_1^{p+1} - x_1'^{p+1}$ is a cycle; hence, by 1.310,

$$([x_1^{p+1} - x_1'^{p+1}] \times z_2^{q-1}) = 0.$$

Then (1.41) follows from (1.22).

Definition 1.42. Suppose that $z_2^{q-1}$ and $\Delta x_1^{p+1} = z_1^p$ are cycles whose bodies are disjoint. The intersection number $(x_1^{p+1} \times z_2^{q-1})$ (which is independent of the choice of $x_1^{p+1}$, provided it is bounded by $z_1^p$) is called the *linking number* of $z_1^p$ and $z_2^{q-1}$ (in the given oriented $R^n$) and is denoted by $\mathfrak{v}(z_1^p, z_2^{q-1})$; it is an element of the coefficient domain $\mathfrak{A}$.

We shall prove the following remarkable and important formula:

$$\mathfrak{v}(z_1^p, z_2^{q-1}) = \pm\mathfrak{v}(z_2^{q-1}, z_1^p), \tag{1.40}$$

or, more precisely,

$$\mathfrak{v}(z_1^p, z_2^{q-1}) = (-1)^{pq+p+1}\mathfrak{v}(z_2^{q-1}, z_1^p). \tag{1.4}$$

Remark. It is of course assumed that both sides of (1.40) and (1.4) are defined, i.e., that if $p = 0$, $z_1^0$ is a normal cycle, and if $q = 1$, $z_2^0$ is a normal cycle.

*Proof.* Choose chains $x^{p+1}$ and $y^q$ in relative general position such that

$$\Delta x^{p+1} = z_1^p, \qquad \Delta y^q = z_2^{q-1}.$$

In view of (1.24) and (1.21) we obtain

$$\begin{aligned}\mathfrak{v}(z_1^p, z_2^{q-1}) &= (x^{p+1} \times \Delta y^q) = (-1)^{p+1}(\Delta x^{p+1} \times y^q)\\ &= (-1)^{p+1}(-1)^{pq}(y^q \times \Delta x^{p+1}) = (-1)^{pq+p+1}\mathfrak{v}(z_2^{q-1}, z_1^p).\end{aligned}$$

This completes the proof.

It now follows that:

If $\mathfrak{v}(z_1^p, z_2^{q-1}) \neq 0$, then $\mathfrak{v}(z_2^{q-1}, z_1^p) \neq 0$; in that case the cycles $z_1^p$, $z_2^{q-1}$ are said to be *linked*. The simplest examples of linking are shown in Figs. 150 and 151, where $z_1^1 = \Delta t_1^2$, $z_2^1 = \Delta t_2^2$ and $\mathfrak{v}(z_1^1, z_2^1)$ is 0 in Fig. 150 and 1 in Fig. 151. In Fig. 152, $z_1^2 = \Delta t_1^3$, $z_2^0 = \Delta t_2^1$, $\mathfrak{v}(z_1^2, z_2^0) = 1$. In Figs. 153, 154, 155 and 156 it is necessary to add mentally properly oriented polygonal lines with sufficiently fine links under the curves (so that the polygonal lines are chains of a complex). The linking numbers are, respec-

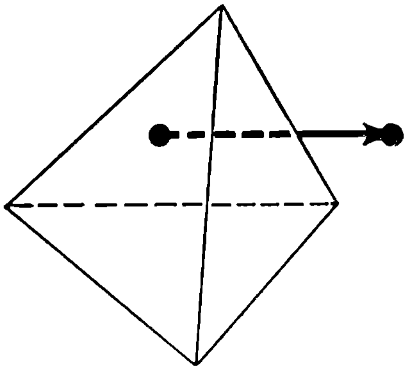


Fig. 152

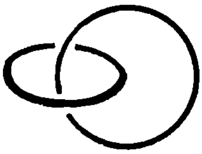


FIG. 153

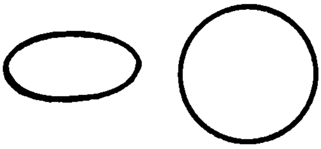


FIG. 154

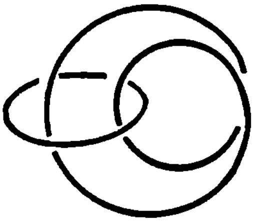


FIG. 155

tively, 1, 0, 2, 0. The reader should in each case verify the validity of (1.40) and (1.4) by constructing the corresponding chains $x_1^2$ and $x_2^2$ bounded by the cycles $z_1^1$ and $z_2^1$. If the ring $\mathfrak{A} = J_2$, then $\mathfrak{v}(z_1^1, z_2^1) = 0$ in Fig. 155.

**§1.5. Computation formulas for the linking numbers.** In this subsection we assume that if $p = 0$, then $z_1^0$, $z_{11}^0$, and $z_{12}^0$ are normal cycles; if $p = n - 1$, $z_2^{q-1}$, $z_{21}^{q-1}$, $z_{22}^{q-1}$ need not be normal cycles.

The formulas

$$(1.51) \qquad \begin{aligned} \mathfrak{v}([z_{11}^p \pm z_{12}^p], z_2^{q-1}) &= \mathfrak{v}(z_{11}^q, z_2^{q-1}) \pm \mathfrak{v}(z_{12}^p, z_2^{q-1}), \\ \mathfrak{v}(z_1^p, [z_{21}^{q-1} \pm z_{22}^{q-1}]) &= \mathfrak{v}(z_1^p, z_{21}^{q-1}) \pm \mathfrak{v}(z_1^p, z_{22}^{q-1}), \end{aligned}$$

$$(1.52) \qquad \mathfrak{v}(az_1^p, z_2^{q-1}) = \mathfrak{v}(z_1^p, az_2^{q-1}) = a\mathfrak{v}(z_1^p, z_2^{q-1})$$

follow from the definition of the linking numbers and from (1.22) and (1.23).

Another consequence is the following important

THEOREM 1.53. *If* $z_1^p \sim 0$ in $R^n \setminus \bar{z}_2^{q-1}$, *then*

$$\mathfrak{v}(z_1^p, z_2^{q-1}) = \mathfrak{v}(z_2^{q-1}, z_1^p) = 0.$$

For, if $z_1^p \sim 0$ in $R^n \setminus \bar{z}_2^{q-1}$, then there exists a chain $x^{p+1} \subset R^n \setminus \bar{z}_2^{q-1}$ such that $\Delta x^{p+1} = z_1^p$ and $(x^{p+1} \times z_2^{q-1}) = 0$.

REMARK 1. Setting

$$z_2^0 = (x^0 - u^0) + (y^0 - v^0)$$

in the plane Fig. 157, where $u^0$, $v^0$ are points with coefficient 1 in the domains exterior to the lemniscate $\bar{z}_1^1$, we have $\mathfrak{v}(z_1^1, z_2^0) = 0$. The cycle $z_1^1$ is linked with each of the normal cycles $x^0 - u^0$, $y^0 - v^0$ and

$$z_1^1 \nsim 0 \quad \text{in } R^2 \setminus \bar{z}_2^0, \qquad z_2^0 \nsim 0 \quad \text{in } R^2 \setminus \bar{z}_1^1.$$

Hence the converse of Theorem 1.53 is not true in general. We obtain the analogous example in three-dimensional space by replacing the pairs of points $x^0 - u^0$, $y^0 - v^0$ by circumferences (1-spheres). The reader should now turn to this example, which is discussed in 3.2 and 3.5 (see the Remark at the end of 3.2 and also Figs. 159, 160).

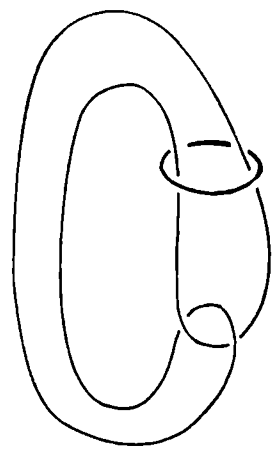


FIG. 156

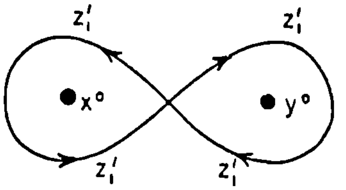


FIG. 157

REMARK 2. In view of Theorem 1.53, it is clear from Fig. 158 that the curves of Fig. 156 are not linked for $\mathfrak{A} = J$. Similarly, by inscribing a Möbius band in $z_1^1$ which does not intersect $\bar{z}_2^1$ [its boundary (mod 2), as in the intuitive geometric sense, will be $z_1^1$] it is verified that the curves of Fig. 155 are not linked for $\mathfrak{A} = J_2$.

In Theorem 1.53, in addition to the coefficient ring $\mathfrak{A}$, from which the coefficients of the cycles $z_1^p$ and $z_2^{q-1}$ are taken, we may also consider still another ring $\mathfrak{A}' \supset \mathfrak{A}$. Then

1.53′. *If a cycle $z_1^p$ is homologous to zero in $R^n \setminus \bar{z}_2^{q-1}$ over the ring $\mathfrak{A}'$, then* $\mathfrak{v}(z_1^p, z_2^{q-1}) = 0$.

This proposition follows immediately from Theorem 1.53 if the cycles $z_1^p$ and $z_2^{q-1}$ are thought of as cycles over $\mathfrak{A}'$ and it is remembered that both $\mathfrak{A}$ and $\mathfrak{A}'$ have the same zero element.

The most important special case is, as always, the case $\mathfrak{A} = J$, $\mathfrak{A}' = \mathfrak{R}$:

1.54. *If $z_1^p$ and $z_2^{q-1}$ are integral cycles and $z_1^p \in \hat{H}^p(R^n \setminus \bar{z}_2^{q-1})$, or if $z_2^{q-1} \in \hat{H}^{q-1}(R^n \setminus \bar{z}^p)$, then* $\mathfrak{v}(z_1^p, z_2^{q-1}) = 0$.

Hence,

1.55. *If $z_{11}^p \sim z_{12}^p$ in $R^n \setminus \bar{z}_2^{q-1}$ over the field of rational numbers $\mathfrak{R}$, then* $\mathfrak{v}(z_{11}^p, z_2^{q-1}) = \mathfrak{v}(z_{12}^p, z_2^{q-1})$.

**§1.6. The case $z^p \in Z_\Delta^p(K_0)$, $*z^{q-1} \in Z_\Delta^{q-1}(K^*)$.** All of the preceding theory remains valid, of course, if $R^n$ is replaced by $S^n$: it is sufficient to

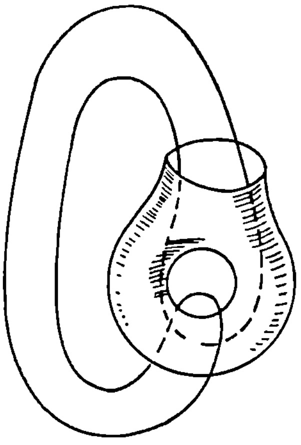


FIG. 158

regard $S^n$ as $R^n$ compactified by the addition of one (infinitely distant) point; it is also possible to develop the entire theory for $S^n$ by beginning all over again and using the terminology of spherical geometry. (In the latter case, however, it would be necessary to restrict the discussion of 0-cycles to normal cycles).

Now let $K$ be a triangulation of $S^n$ and let $K_1$ and $K^*$ be respectively the barycentric subdivision and the complex of the barycentric stars (the barycentric complex) of $K$ (IV, 5.2). For the sequel, we choose a definite orientation of $K$.

Suppose that $z^p \in Z_\Delta^p(K)$, ${}^*z^{q-1} \in Z_\Delta^{q-1}(K^*)$. Then the linking number $\mathfrak{v}(z^p, {}^*z^{q-1})$ of XIII, Def. 4.5 is defined. On the other hand, we have just defined the linking number $\mathfrak{v}(z^p, s_1{}^*z^{q-1})$ of the cycle $z^p$ and the barycentric subdivision $s_1{}^*z^{q-1}$ of the cycle ${}^*z^{q-1}$ (the conditions of 1.4 are obviously satisfied). We shall prove that

$$(1.61) \qquad \mathfrak{v}(z^p, s_1{}^*z^{q-1}) = \mathfrak{v}(z^p, {}^*z^{q-1}).$$

Returning to the notation of XIII, we denote by $t^n$ the orientations of the $n$-simplexes $T^n \in K$ coherent with the chosen orientation of $K$. Let $t^p$, $p < n$, be arbitrary orientations of the simplexes $T_i^p \in K_1$ and let ${}^*t_i^q$ be the barycentric star dual to $T_i^p$ oriented so that $(t_i^p \times {}^*t_i^q) = 1$ in the sense of XIII, 3.1. From this definition it follows, in particular, that ${}^*t_i^0$ is the centroid of $T_i^n$ taken with coefficient $+1$.

The chains $t_i^p$ and ${}^*t_h^q$ satisfy the conditions (1.32); hence the intersection number of these chains is also defined in the sense of 1.33, and we denote this intersection number (until the end of this section) by $(t_i^p \otimes {}^*t_h^q)$. We shall prove the lemma

$$(1.62) \qquad (t_i^p \otimes {}^*t_h^q) = (t_i^p \times {}^*t_h^q).$$

Since the bodies of $t_i^p$ and $*t_l^q$ $(i \neq l)$ are disjoint and $(t_i^p \times *t_i^q) = 1$, it is sufficient to show

$$(1.63) \qquad (t_i^p \otimes *t_i^q) = 1.$$

It is easily seen that (1.63) is true for $q = 0$, $p = n$. Let us assume that it has been proved for $q$ and prove it for $q + 1$, i.e., we assume that

$$(t_i^p \otimes *t_i^q) = 1$$

and prove that

$$(t_j^{p-1} \otimes *t_j^{q+1}) = 1.$$

To this end, suppose that $|t_i^p| \in O_K|t_j^{p-1}|$; then $(t_i^p:t_j^{p-1}) \neq 0$ and according to XIII, (3.31),

$$(*t_j^{q+1}:*t_i^q) = (-1)^p(t_i^p:t_j^{p-1}).$$

Applying (1.24), we obtain

$$\begin{aligned}(t_j^{p-1} \otimes *t_j^{q+1}) &= (t_i^p:t_j^{p-1})(\Delta t_i^p \otimes *t_j^{q+1})\\ &= (-1)^p(t_i^p:t_j^{p-1})(t_i^p \otimes \Delta *t_j^{q+1})\\ &= (-1)^p(t_i^p:t_j^{p-1})(*t_j^{q+1}:*t_i^q)(t_i^p \otimes *t_i^q)\\ &= (-1)^p(-1)^p(t_i^p:t_j^{p-1})(t_i^p:t_j^{p-1})(t_i^p \otimes t_i^q)\\ &= 1,\end{aligned}$$

which completes the proof.

We now denote by $x^{p+1} = \sum c_h t_h^{p+1}$ a chain of $K$ bounded by the cycle $z^p$.

Suppose that

$$*z^{q-1} = \sum d_h *t_h^{q-1} \in L^{q-1}(K^*).$$

On the one hand (by XIII, Def. 4.5),

$$\mathfrak{v}(z^p, *z^{q-1}) = (x^{p+1} \times *z^{q-1}) = \sum_{i,h} c_i d_h (t_i^{p+1} \times *t_h^{q-1});$$

while, on the other hand, the linking number of these cycles in the sense of Def. 1.42 is equal to $\sum_{i,h} c_i d_h (t_i^{p+1} \otimes *t_h^{q-1})$. In view of (1.62), both expressions are equal and this completes the proof.

## §2. Linking of proper cycles

**§2.1. Definition of linking numbers for proper cycles.** Suppose that $\Phi \subset R^n$. Every $\epsilon$-cycle $z^p$ of the compactum $\Phi$ is a cycle of the open set $S(\Phi, \epsilon)$ (see the introductory remarks at the beginning of this chapter). Hence if $z^{q-1}$ is a cycle of the open set $R^n \setminus \bar{S}(\Phi, \epsilon)$, the linking number $\mathfrak{v}(z^p, z^{q-1})$ is defined.

Now let

$$\mathfrak{z}^p = (z_1^p, z_2^p, \cdots, z_k^p, \cdots)$$

be a proper cycle of $\Phi$ and let $z^{q-1} \in Z^{q-1}(\Gamma)$ be a cycle of the open set $\Gamma = R^n \setminus \Phi$. There is an $\epsilon > 0$ for which $z^{q-1}$ is a cycle of the open set $\Gamma_\epsilon = R^n \setminus \bar{S}(\Phi, \epsilon)$. On the other hand, the definition of proper cycle implies that all the $z_k^p$ for sufficiently large $k$ are cycles of the open set $S(\Phi, \epsilon)$ and are homologous (to each other) in this open set. Hence, in view of Theorem 1.53, the linking number $\mathfrak{v}(z_k^p, z^{q-1})$ has the same value for all sufficiently large $k$. This constant value is called *the linking number* $\mathfrak{v}(\mathfrak{z}^p, z^{q-1})$ *of the proper cycle* $\mathfrak{z}^p$ *of the compactum* $\Phi$ *and the cycle* $z^{q-1}$ *of the open set* $\Gamma = R^n \setminus \Phi$:

$$\mathfrak{v}(\mathfrak{z}^p, z^{q-1}) = \mathfrak{v}(z_k^p, z^{q-1}) \tag{2.10}$$

for sufficiently large $k$.

Now let $\Phi$ and $\Phi'$ be two disjoint closed subsets of $R^n$. Let $\epsilon > 0$ be such that the neighborhoods $S(\Phi, \epsilon)$ and $S(\Phi', \epsilon)$ are disjoint. Let

$$\mathfrak{z}^p = (z_1^p, z_2^p, \cdots, z_k^p, \cdots)$$

and

$$\mathfrak{z}^{q-1} = (z_1^{q-1}, z_2^{q-1}, \cdots, z_k^{q-1}, \cdots)$$

be proper cycles of $\Phi$ and $\Phi'$, respectively. For all sufficiently large $k$ the cycles $z_k^p$ are in $S(\Phi, \epsilon)$ and are homologous there, and the cycles $z_k^{q-1}$ are in $S(\Phi', \epsilon)$ and are also homologous in that set. Therefore, the linking number $\mathfrak{v}(z_k^p, z_k^{q-1})$ has the same value for all sufficiently large $k$. This value is called *the linking number* $\mathfrak{v}(\mathfrak{z}^p, \mathfrak{z}^{q-1})$ *of the proper cycles* $\mathfrak{z}^p$ *and* $\mathfrak{z}^{q-1}$.

REMARK 1. It is clear that

$$\mathfrak{v}(\mathfrak{z}^p, \mathfrak{z}^{q-1}) = \mathfrak{v}(z_i^p, z_j^{q-1})$$

for all sufficiently large $i$ and $j$.

From these definitions it now easily follows that (1.51) and (1.52) remain valid for the linking numbers of proper cycles of $\Phi$ and cycles of $\Gamma = R^n \setminus \Phi$, and also for the linking numbers of proper cycles contained, respectively, in disjoint compacta $\Phi$ and $\Phi'$.

Furthermore,

2.11. *If*

$$z^{q-1} \sim 0 \quad in\ \Gamma$$

*or*

$$\mathfrak{z}^p \sim 0 \quad in\ \Phi,$$

*then*

$$\mathfrak{v}(\mathfrak{z}^p, z^{q-1}) = 0.$$

*In exactly the same way, if at least one of the conditions*

$$\mathfrak{z}^p \sim 0 \quad \textit{in } \Phi, \qquad \mathfrak{z}^{q-1} \sim 0 \quad \textit{in } \Phi'$$

*holds, then*

$$\mathfrak{v}(\mathfrak{z}^p, \mathfrak{z}^{q-1}) = 0,$$

*where* $\mathfrak{z}^p$ *and* $\mathfrak{z}^{q-1}$ *are proper cycles contained respectively in disjoint compacta* $\Phi$ *and* $\Phi'$.

It follows that

2.111. If $\mathfrak{z}_1^p$, $\mathfrak{z}_2^p$ are proper cycles of $\Phi$ which are homologous in $\Phi$, and $\mathfrak{z}^{q-1}$ is a proper cycle of $\Phi' \subset R^n \setminus \Phi$, then

$$\mathfrak{v}(\mathfrak{z}_1^p, \mathfrak{z}^{q-1}) = \mathfrak{v}(\mathfrak{z}_2^p, \mathfrak{z}^{q-1}).$$

DEFINITION 2.120. *Two systems of cycles*

$$\mathfrak{z}_1^p, \cdots, \mathfrak{z}_s^p \quad \textit{in } \Phi \tag{2.121}$$

*and*

$$z_1^{q-1}, \cdots, z_s^{q-1} \quad \textit{in } \Gamma \tag{2.122}$$

*are said to be linked if*

$$\mathfrak{v}(\mathfrak{z}_i^p, z_j^{q-1}) = \delta_{ij} \qquad (i, j = 1, 2, \cdots, s).$$

In this definition the cycles (2.122) may be replaced by proper cycles of any compactum $\Phi' \subset \Gamma = R^n \setminus \Phi$.

THEOREM 2.12. *If the systems* (2.121) *and* (2.122) *are linked, then the cycles of* (2.121) *are lirh in* $\Phi$ *and the cycles of* (2.122) *are lirh in* $\Gamma$.

The proofs of both assertions are completely identical.

For instance, suppose that $\sum c_i\mathfrak{z}_i^p \sim 0$ in $\Phi$. Then by 2.11,

$$\mathfrak{v}(\sum c_i\mathfrak{z}_i^p, z_h^{q-1}) = 0 \qquad (1 \le h \le s).$$

But

$$\mathfrak{v}(\sum c_i\mathfrak{z}_i^p, z_h^{q-1}) = \sum c_i\mathfrak{v}(\mathfrak{z}_i^p, z_h^{q-1}) = \sum_i c_i\delta_{ih} = c_h.$$

Hence $c_h = 0$ $(1 \le h \le s)$, which completes the proof.

**§2.2. A deformation theorem for singular cycles.**

2.2. *Let* $\mathfrak{z}_0^p$ *and* $\mathfrak{z}_0^{q-1}$ *be two singular cycles in* $R^n$, *and suppose that their parametric representations* (XI, 4.7) *are*

$$\mathfrak{z}_0^p = (C_0z_1^p, C_0z_2^p, \cdots)$$

*and*

$$\mathfrak{z}_0^{q-1} = (C_0' z_1^{q-1}, C_0' z_2^{q-1}, \cdots).$$

*If*

$$\left.\begin{aligned} \mathfrak{z}_\theta^p &= (C_\theta z_1^p, C_\theta z_2^p, \cdots),\\ \mathfrak{z}_\theta^{q-1} &= (C_\theta' z_1^{q-1}, C_\theta z_2^{q-1}, \cdots) \end{aligned}\right\} \qquad (0 \leq \theta \leq 1)$$

*are deformations of these parametric representations, where for each value of* $\theta$ *there are disjoint compacta* $\Phi_\theta$ *and* $\Phi_\theta'$ *in* $R^n$ *which carry the singular cycles* $\mathfrak{z}_\theta^p$ *and* $\mathfrak{z}_\theta^{q-1}$ *respectively, then*

$$\mathfrak{v}(\mathfrak{z}_0^p, \mathfrak{z}_0^{q-1}) = \mathfrak{v}(\mathfrak{z}_1^p, \mathfrak{z}_1^{q-1}).$$

*Proof.* For a prescribed $\theta_0$, $0 \leq \theta_0 \leq 1$, let $S(\Phi_{\theta_0})$, $S(\Phi_{\theta_0}')$ be disjoint neighborhoods of $\Phi_{\theta_0}$ and $\Phi_{\theta_0}'$, and let $U_{\theta_0}$ be a neighborhood of $\theta_0$ relative to the segment $0 \leq \theta \leq 1$ such that $\theta \in U_{\theta_0}$ implies that $\Phi_\theta \subseteq S(\Phi_{\theta_0})$ and $\Phi_\theta' \subseteq S(\Phi_{\theta_0}')$. Then, according to XI, Theorem 3.42, $\mathfrak{z}_\theta^p \sim \mathfrak{z}_{\theta_0}^p$ in $S(\Phi_{\theta_0})$ and $\mathfrak{z}_\theta^{q-1} \sim \mathfrak{z}_{\theta_0}^{q-1}$ in $S(\Phi_{\theta_0}')$. Therefore, in view of the results of 2.1, $\mathfrak{v}(\mathfrak{z}_\theta^p, \mathfrak{z}_\theta^{q-1})$ is constant in $U_{\theta_0}$. Since this is true for arbitrary $\theta_0$, $\mathfrak{v}(\mathfrak{z}_\theta^p, \mathfrak{z}_\theta^{q-1})$ is constant on $0 \leq \theta \leq 1$ and consequently $\mathfrak{v}(\mathfrak{z}_0^p, \mathfrak{z}_0^{q-1}) = \mathfrak{v}(\mathfrak{z}_1^p, \mathfrak{z}_1^{q-1})$. This completes the proof.

REMARK. The sphere $S^n$ may be substituted for the space $R^n$ in all the arguments of this section.

## §3. The little Alexander duality

**§3.1. Introductory remarks and formulation of the theorem.** In this section we use the following notation: $\Phi$ is a curved polyhedron imbedded in $S^n$; $\Gamma = S^n \setminus \Phi$; $K_0$ is a curved triangulation of $\Phi$; $\mathfrak{A}$ is the field of rational numbers or a field $J_m$, $m$ a prime; $\pi^r(\Phi, \mathfrak{A})$, $\pi^r(\Gamma, \mathfrak{A})$ denote the $r$th Betti numbers (if $\mathfrak{A} = \mathfrak{R}$) and the $r$th Betti numbers (mod $m$) (if $\mathfrak{A} = J_m$) of $\Phi$ and $\Gamma$, respectively; in this connection, $\pi^0(\Phi, \mathfrak{A})$ or $\pi^0(\Gamma, \mathfrak{A})$ is defined as the number of components of $\Phi$ or $\Gamma$ less 1, whatever $\mathfrak{A}$ is. In this notation, the *little Alexander duality* is expressed by the formula

$$\pi^p(\Phi, \mathfrak{A}) = \pi^{q-1}(\Gamma, \mathfrak{A}), \tag{3.1}$$

with $0 \leq p \leq n - 1$, $q = n - p$.

We shall give an elementary proof of the little Alexander duality in this section. The proof is based on the results of XIII, but is completely independent of those of XIV.

Let $Q$ be a fixed triangulation of $S^n$ and let $Q_\epsilon$ be a subdivision of $Q$ of mesh $< \epsilon$; we then denote by $K_\epsilon$ the complex consisting of all the simplexes of $Q_\epsilon$ whose closures intersect $\Phi$ together with all the faces of these simplexes and by $\Phi_\epsilon$ the body of the triangulation $K_\epsilon$.

**§3.2. Proof of Theorem (3.1).**

LEMMA 3.20. *If $3\epsilon$ is a Lebesgue number of $K_0$, then a cycle $z^p \in Z^p(K_0, \mathfrak{A})$ which is not homologous to zero in $K_0$ is not homologous to zero in $\Phi_\epsilon$.*

*Proof.* Suppose that $z^p \in Z^p(K_0, \mathfrak{A})$, regarded as a proper cycle (see XI, 4.6), is homologous to zero in $\Phi_\epsilon$.

Let $h$ be large enough so that the barycentric subdivision $s_h z^p$ of $z^p$ of order $h$ is the boundary of an $\epsilon$-chain $x_\epsilon$ of $\Phi_\epsilon$:

$$\Delta x_\epsilon = s_h z^p.$$

Assign to each vertex $e_i$ of $x_\epsilon$ not contained in $\Phi$ a point of $\Phi$ which is near $e_i$; this assignment defines an $\epsilon$-displacement which maps $x_\epsilon$ into a $3\epsilon$-chain $x_{3\epsilon}$ of $\Phi$, and

$$\Delta x_{3\epsilon} = s_h z^p.$$

A canonical displacement relative to $K_0$ now maps the chain $x_{3\epsilon}$ into a chain $x_0$ of $K_0$ and $s_h z^p$ is transformed into $z^p = \Delta x_0$.

A special case of Lemma 3.20 is

LEMMA 3.21. If the cycles

$$(3.21) \qquad z_1^p, \cdots, z_s^p \qquad [z_i^p \in Z^p(K_0, \mathfrak{A})]$$

are lirh in $K_0$ and $3\epsilon$ is a Lebesgue number of $K_0$, then the system (3.21) is lirh in $\Phi_\epsilon$.

LEMMA 3.22. If the cycles (3.21) are lirh in $K_0$, then there is a system of cycles

$$(3.22) \qquad z_1^{q-1}, \cdots, z_s^{q-1} \qquad [z_i^{q-1} \in Z^{q-1}(\Gamma, \mathfrak{A})]$$

in $\Gamma$ which is linked with the system (3.21); consequently, the system (3.22) is lirh in $\Gamma$.

*Proof of Lemma* 3.22. In virtue of Lemma 3.21, the cycles (3.21) (regarded as proper cycles of $\Phi_\epsilon$) are lirh in $\Phi_\epsilon$; a canonical displacement of $\Phi_\epsilon$ into $K_\epsilon$ transforms them into cycles

$$(3.210) \qquad z_{01}^p, \cdots, z_{0s}^p$$

of $K_\epsilon$, lirh in $K_\epsilon$. In view of XIII, Theorem 4.61 we can construct a system of cycles (3.22) in $\Gamma_\epsilon = S^n \setminus \Phi_\epsilon$ which is linked with the system (3.210). Since $z_i^p \sim z_{0i}^p$ in $\Phi_\epsilon$,

$$\mathfrak{v}(z_i^p, z_j^{q-1}) = \mathfrak{v}(z_{0i}^p, z_j^{q-1});$$

hence (3.21) and (3.22) are linked.

REMARK. Suppose that the cycles (3.21) are integral cycles of $K_0$ which form a $(J, \mathfrak{N})$-basis in $K_0$. Repeating the preceding argument, but this time using XIII, Theorem 4.62 (instead of Theorem 5.61) at the end, we obtain a system (3.22) of integral cycles linked with (3.21).

Hence

3.220. *If a system of integral cycles* (3.21) *forms a* $(J, \mathfrak{N})$*-basis in* $K_0$, *then there exists a system* (3.22) *of integral cycles linked with* (3.21).

We shall return to this proposition in 3.3.

LEMMA 3.23. If (3.22) is a system of cycles lirh in $\Gamma$, then there exists a system of cycles (3.21) lirh in $K_0$ which is linked with the system (3.22).

*Proof.* There is an $\epsilon > 0$ for which all the cycles (3.22) are contained in the polyhedron $\bar{\Gamma}_\epsilon$, where $\Gamma_\epsilon = S^n \setminus \Phi_\epsilon$, and are lirh in $\bar{\Gamma}_\epsilon$. In view of the preceding lemma, there exist cycles

$$z_1'^p, \cdots, z_s'^p$$

in $S^n \setminus \bar{\Gamma}_\epsilon \subseteq \Phi_\epsilon$ such that

$$\mathfrak{v}(z_i'^p, z_j^{q-1}) = \delta_{ij}.$$

Replacing the cycles $z_i'^p$ by sufficiently fine subdivisions of these cycles, we obtain, after a canonical displacement, the required cycles

$$z_1^p, \cdots, z_s^p.$$

This completes the proof of Lemma 3.23.

Now let (3.21) be a maximal system of cycles lirh in $K_0$. Then

$$\pi^p(\Phi, \mathfrak{A}) = \pi^p(K_0, \mathfrak{A}) = s.$$

To show this, we construct a system of cycles (3.22) linked with (3.21). The cycles (3.22) are lirh in $\Gamma$. We shall show that they form a *maximal* system of cycles lirh in $\Gamma$.

Suppose that there is a cycle $z^{q-1} \in Z^{q-1}(\Gamma, \mathfrak{A})$ such that

$$z^{q-1}, z_1^{q-1}, \cdots, z_s^{q-1}$$

are lirh in $\Gamma$. Then, by Lemma 3.23, there exist $s + 1$ lirh $p$-cycles, which contradicts the fact that $\pi^p(K_0, \mathfrak{A}) = s$. Since (3.22) is a maximal system,

$$\pi^{q-1}(\Gamma, \mathfrak{A}) = s = \pi^p(\Phi, \mathfrak{A}).$$

This completes the proof.

REMARK. It can be proved in exactly the same way that every system of cycles (3.21) which is linked with a maximal system of cycles (3.22) of the open set $\Gamma$ is also maximal.

Hence,

3.2. *Let* $\Phi = \| K_0 \|$ *be a curved polyhedron in* $S^n$. *Given a maximal system of lirh p-cycles of* $K_0$ (*or of* $\Gamma = S^n \setminus \Phi$), *it is possible to construct a system*

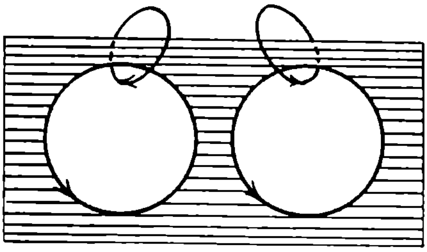


FIG. 159

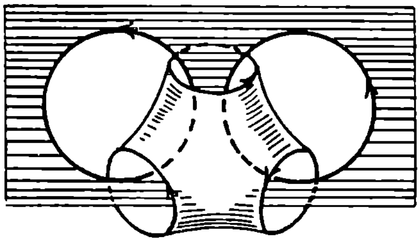


FIG. 160

*of* $(q - 1)$*-cycles of* $\Gamma$ *(or of* $K_0$*) linked with it; furthermore, the resulting system is maximal.*

In Theorem 3.2 the coefficient domain can be either the field of rational numbers or any of the fields $J_m$, $m$ a prime. Fig. 159, where $p = q - 1 = 1$, $n = 3$ and the polyhedron $\Phi$ is hatched, illustrates the theorem.

REMARK. The cycles $z_1^p$ and $z_2^p$ of Fig. 160 form a $(J, \mathfrak{R})$-basis in $\Phi$ and the cycles $z_1^{q-1}$, $z_2^{q-1}$ form a $(J, \mathfrak{R})$-basis in $\Gamma$ (Fig. 160 shows the homology $z^{q-1} \sim z_1^{q-1} + z_2^{q-1}$ in $\Gamma$). Therefore, Figs. 159 and 160 exemplify not only Theorem 3.2 but the so called little Pontryagin duality as well. We now turn to this duality.

**§3.3. The little Pontryagin duality.** Since the concept of homology basis coincides with that of a maximal system of lirh cycles of a given dimension for the coefficient domains prescribed at the beginning of this section, Theorem 3.2 may be reformulated as

3.30. *Given a p-dimensional homology basis of* $K_0$ *(or* $\Gamma$*) over* $\mathfrak{A}$*, it is possible to construct a* $(q - 1)$*-dimensional basis of* $\Gamma$ *(or* $K_0$*) linked with the prescribed system.*

This is the theorem we refer to as the little Pontryagin duality over the coefficient domains $\mathfrak{A} = J_m$, $\mathfrak{A} = \mathfrak{R}$.

A deeper result is the analogous proposition for the $(J, \mathfrak{R})$-bases:

3.3. THE LITTLE PONTRYAGIN DUALITY. *Given a p-dimensional* $(J, \mathfrak{R})$*-basis of* $K_0$ *(or of the open set* $\Gamma$*), it is possible to construct a* $(q - 1)$*-dimensional* $(J, \mathfrak{R})$*-basis of the open set* $\Gamma$ *(or of the complex* $K_0$*) which is linked with the prescribed basis.*

*Proof.* We know from Theorems 3.220 and 3.2 that, given a $p$-dimensional $(J, \mathfrak{R})$-basis

$$z_1^p, \cdots, z_\pi^p \tag{3.31}$$

of $K_0$, it is possible to construct a maximal system of integral cycles

$$z_1^{q-1}, \cdots, z_\pi^{q-1} \tag{3.32}$$

of $\Gamma$ linked with (3.31). We shall prove that (3.32) is a $(J, \mathfrak{R})$-basis for $\Gamma$, i.e., that for every integral $(q-1)$-cycle $z^{q-1}$ there exists a linear combination $\sum c_j z_j^{q-1}$ of the cycles (3.32) with integral coefficients $c_j$ such that

$$z^{q-1} - \sum_j c_j z_j^{q-1} \in \hat{H}^{q-1}(\Gamma).$$

First, we note that because the system (3.32) is maximal,

$$z^{q-1} \;(\Gamma\text{-}\sim)\; \sum_j c_j z_j^{q-1},$$

where the coefficients $c_j$ are rational.

Hence

$$\mathfrak{v}(z_i^p, z^{q-1}) = \sum_j c_j \mathfrak{v}(z_i^p, z_j^{q-1}) = \sum_j c_j \delta_{ij} = c_i .$$

Since $\mathfrak{v}(z_i^p, z^{q-1})$ is an integer, all the coefficients $c_i$ are integers, and so

$$z^{q-1} - \sum_i c_i z_i^{q-1}$$

is an integral cycle. Since it is homologous to zero over $\mathfrak{R}$, it is contained in $\hat{H}^{q-1}(\Gamma)$. This proves that the cycles (3.32) form a $(J, \mathfrak{R})$-basis for $\Gamma$.

Remark. This proves, in particular, that $\Gamma$ has a $(J, \mathfrak{R})$-basis for every dimension $r$.

Now suppose that (3.32) is any $(J, \mathfrak{R})$-basis of $\Gamma$.

Let (3.31) be any $(J, \mathfrak{R})$-basis of $K_0$ and construct a $(J, \mathfrak{R})$-basis

$$u_1^{q-1}, \cdots, u_\pi^{q-1} \tag{3.33}$$

of $\Gamma$ linked with it. Since (3.32) and (3.33) are $(J, \mathfrak{R})$-bases of $\Gamma$,

$$z_h^{q-1} \;(\Gamma\text{-}\sim)\; \sum_k a_{hk} u_k^{q-1},$$

$$u_k^{q-1} \;(\Gamma\text{-}\sim)\; \sum_h b_{kh} z_h^{q-1},$$

where the matrices $A = (a_{hk})$ and $B = (b_{kh})$ have integral elements and are inverse to each other; consequently, $A$ and $B$ are unimodular.

We now seek linear combinations

$$u_i^p = \sum_j \xi_{ij} z_j^p$$

with integral coefficients of the cycles $z_j^p$ such that

$$\mathfrak{v}(u_i^p, z_h^{q-1}) = \delta_{ih} .$$

To this end, we solve the system of equations

$$\mathfrak{v}\left(\textstyle\sum_j \xi_{ij} z_j^p,\ z_h^{q-1}\right) = \delta_{ih} \qquad [i, h = 1, 2, \cdots, \pi^p(K_0) = \pi^{q-1}(\Gamma)]$$

for the $\xi_{ih}$. But

$$z_h^{q-1} \ (\Gamma\text{-}\sim) \ \textstyle\sum_k a_{hk} u_k^{q-1}.$$

Therefore, the system of equations may be rewritten as

$$\mathfrak{v}\left(\textstyle\sum_j \xi_{ij} z_i^p,\ \sum_k a_{hk} u_k^{q-1}\right) = \delta_{ih},$$

i.e.,

$$\textstyle\sum_{k,j} a_{hk} \xi_{ij} \mathfrak{v}(z_j^p,\ u_k^{q-1}) = \delta_{ih};$$

or, since $\mathfrak{v}(z_j^p, u_k^{q-1}) = \delta_{jk}$, as

$$\textstyle\sum a_{hj} \xi_{ij} = \delta_{ih}.$$

Since det $(a_{ki}) = \pm 1$, this system has a unique integral solution.

The resulting integral cycles $u_i^p$ form a maximal system of lirh integral cycles of $K_0$. Exactly the same reasoning which we used in the first part of the proof will show that the cycles $u_i^p$ form a $(J, \mathfrak{R})$-basis of $K_0$. This completes the proof of the theorem.

**§3.4. Closed $(n - 1)$-dimensional pseudomanifolds in $S^n$ $(n \geq 2)$. The Jordan-Brouwer theorem.** Since $\pi^0(\Gamma, \mathfrak{A}) + 1$ (for every $\mathfrak{A}$) is equal to the number of components of $\Gamma$, we deduce from (3.11) (setting $p = n - 1$ and $\mathfrak{A} = J_m$):

3.41. *For every curved polyhedron $\Phi$ in $S^n$,*

$$\pi^{n-1}(\Phi) = \pi_m^{n-1}(\Phi).$$

If, in particular, $\Phi$ is any $(n - 1)$-dimensional closed pseudomanifold, then $\pi_2^{n-1}(\Phi) = 1$; hence $\pi^{n-1}(\Phi) = 1$, and $\Phi$ is orientable. Since $\pi^0(\Gamma) = 1$, $\Gamma$ consists of two components.

Therefore,

3.411. *Every $(n - 1)$-dimensional closed pseudomanifold in $S^n$ is orientable and separates $S^n$ into two domains.*

We shall now prove that if $\Phi'$ is an arbitrary closed proper subset of $\Phi$, then the open set $\Gamma' = R^n \setminus \Phi'$ is connected. To do so, it is sufficient to show that every pair of points $p_1$ and $p_2$ of $\Gamma'$ is contained in a connected subset of $\Gamma'$. To this end, we take a curved triangulation $K$ of $\Phi$ of sufficiently fine mesh such that the subcomplex $K_0$ of $K$ consisting of all the simplexes of $K$ which intersect $\Phi'$ and their faces is a proper subcomplex of $K$ whose body $\| K_0 \|$ contains neither of the points $p_1$, $p_2$.

Then $\pi^{n-1}(K_0) = 0$, so that in virtue of (3.1) the open set

$$\Gamma_0 = R^n \setminus \| K_0 \| \subseteq \Gamma'$$

is connected. Since $\Gamma_0$ contains both of the points $p_1$ and $p_2$, we have proved our assertion.

It now follows that (see II, Theorem 1.2):

3.42. THE JORDAN-BROUWER THEOREM. *Every* $(n-1)$*-dimensional closed pseudomanifold in* $S^n$ *is orientable, divides* $S^n$ *into two domains and is the common boundary of both domains.*

This theorem obviously remains true if $R^n$ is substituted for $S^n$: it is enough to map $S^n$ onto $R^n$ by means of a stereographic projection with center of projection lying outside the given pseudomanifold $\Phi$.

Let us consider more closely the case of a closed $(n-1)$-dimensional pseudomanifold $\Phi$ in $R^n$. The bounded component of the open set $\Gamma = R^n \setminus \Phi$ is called the *interior* of $\Phi$ and will be denoted by $\Gamma_0$; the unbounded component of $\Gamma$ is called the *exterior* of $\Phi$ and we shall denote it by $\Gamma_\infty$.

Let $z^{n-1}$ be an orientation of the pseudomanifold $\Phi$.

If $o' \in \Gamma_\infty$, then $\mathfrak{v}(z^{n-1}, o') = 0$. For, since $o' \sim o''$ in $\Gamma_\infty$ for two arbitrary points of $\Gamma_\infty$, it follows that $\mathfrak{v}(z^{n-1}, o')$ assumes the same value for all $o' \in \Gamma_\infty$. To see that this value is zero, it is sufficient to take $o'$ exterior to a cube $Q^n$ containing $\Phi$ in its interior: then $z^{n-1} \sim 0$ in $Q^n$ and consequently

$$\mathfrak{v}(z^{n-1}, o') = 0$$

for $o' \in R^n \setminus Q^n$.

To determine $\mathfrak{v}(z^{n-1}, o)$, where $o \in \Gamma$, we first note that by Theorem 3.220 there exists an integral 0-cycle

$$z^0 = \sum a_i o_i + \sum a_i' o_i' \qquad (o_i \in \Gamma_0\,,\; o_i' \in \Gamma_\infty)$$

such that

$$\mathfrak{v}(z^{n-1}, z^0) = 1.$$

Since $\mathfrak{v}(z^{n-1}, o_i') = 0$ and $\sum a_i o_i \sim (\sum a_i) o$, where $o$ is any point of $\Gamma_0$, it follows that

$$\mathfrak{v}(z^{n-1}, ao) = 1,$$

where $a = \sum a_i$. Since

$$\mathfrak{v}(z^{n-1}, ao) = a\mathfrak{v}(z^{n-1}, o)$$

and $a$ is an integer,

$$\mathfrak{v}(z^{n-1}, o) = \pm 1.$$

Hence *the linking number* $\mathfrak{v}(z^{n-1}, o)$ *assumes the same value,* 1 *or* $-1$, *for all points* $o \in \Gamma_0$.

REMARK 1. We shall prove the last assertion using Lemma 3.22 instead of

Theorem 3.220. By 3.22 there exists a 0-cycle $z^0$ over $\mathfrak{A}$ for which $\mathfrak{v}(z^{n-1}, z^0) = 1$, that is, there is an integral cycle $z^0$ for which $\mathfrak{v}(z^{n-1}, z^0) \neq 0$. Repeating the above argument, we see that $\mathfrak{v}(z^{n-1}, ao) \neq 0$, so that $\mathfrak{v}(z^{n-1}, o) \neq 0$. This result remains valid if the coefficient domain $J$ is replaced by $J_m$, $m$ a prime. Hence

$$\mathfrak{v}(z^{n-1}, o) \not\equiv 0 \pmod{m},$$

where $m$ is an arbitrary prime. But it then follows that $\mathfrak{v}(z^{n-1}, o) = \pm 1$.

We may therefore state the following theorem:

3.43. *If $\Phi \subset R^n$ is an $(n - 1)$-dimensional closed pseudomanifold and $z^{n-1}$ is an orientation of $\Phi$, then the linking number $\mathfrak{v}(z^{n-1}, o)$ assumes the same value at all points of each of the two domains into which $\Phi$ divides $R^n$; this value is zero on the exterior domain and $\pm 1$ on the interior domain.*

Definition 3.44. The number $\mathfrak{v}(z^{n-1}, o)$, where $o \in R^n \setminus \Phi$, is called *the index of the oriented pseudomanifold* $\Phi$ relative to the point $o$.

§**3.5. Remarks on linking of pseudomanifolds.** In this subsection all cycles are integral cycles and all homologies are weak homologies (that is, homologies over $\mathfrak{R}$). In 1.5, Remark 1 we gave an example of a cycle $z^p$ (whose body was the polyhedron $\Phi$) and a cycle $z^{q-1}$ of the open set $\Gamma = R^n \setminus \Phi$, which was not homologous to zero in $\Gamma$, but was not linked with $z^p$. Now let $\Phi$ be a closed orientable $p$-dimensional pseudomanifold in $R^n$, let $z^p$ be an orientation of $\Phi$ and let $z^{q-1}$ be a cycle of the open set $\Gamma = R^n \setminus \Phi$, which is not homologous to zero in $\Gamma$. By Theorem 3.3, there exists in $\Gamma$ a cycle $z_1^{q-1}$ forming a $(J, \mathfrak{R})$-basis in $\Gamma$ and such that $\mathfrak{v}(z^p, z_1^{q-1}) = 1$. Then $z^{q-1} \sim cz_1^{q-1}$ in $\Gamma$. Since $z^{q-1} \not\sim 0$ in $\Gamma$, it follows that $c \neq 0$; hence,

$$\mathfrak{v}(z^p, z^{q-1}) = c\mathfrak{v}(z^p, z_1^{q-1}) = c \neq 0.$$

Therefore,

3.51. *If $\Phi$ is a closed orientable $p$-dimensional pseudomanifold and $z^{q-1}$ is a $(q - 1)$-cycle in $\Gamma = R^n \setminus \Phi$, then $z^{q-1} \sim 0$ in $\Gamma$ if, and only if, it is not linked with $\Phi$ [that is, if and only if $\mathfrak{v}(z^p, z^{q-1}) = 0$ for an arbitrary orientation $z^p$ of $\Phi$].*

As a special case, let $\Phi$ and $\Phi'$ be two disjoint closed orientable pseudomanifolds of dimensions $p$ and $q - 1$ in $R^n$. If one of these pseudomanifolds is homologous to zero in the complement of the second, then the second is also homologous to zero in the complement of the first, and this can occur if, and only if, the pseudomanifolds are not linked. [In this context, homology and linking of pseudomanifolds refer, of course, to (arbitrary) orientations of the pseudomanifolds, and not to the pseudomanifolds themselves.] We define *the linking number of two disjoint closed orientable pseudomanifolds $\Phi$ and $\Phi'$ of dimensions $p$ and $q - 1$ in $R^n$ to be the absolute value of the linking number of arbitrary orientations of the pseudomanifolds.* The nonnegative integer obtained in this way expresses a certain property of the mutual imbedding of the pseudomanifolds $\Phi$ and $\Phi'$ in $R^n$; the vanishing or nonvanishing of this number corresponds to the intuitive notion of "nonlinking" or "linking" (of two simple closed curves in $R^3$, for example). We shall prove that the linking number of two pseudomanifolds is indeed a *positional topological invariant*, that is, a property which depends on the mutual imbedding of the two pseudomanifolds, but is invariant under topological mappings of the *whole* space $R^n$ in which both pseudomanifolds are imbedded. In other words, we shall prove

3.52. *If $C$ is a topological mapping of $R^n$ onto itself or onto another Euclidean space $R_1{}^n$, then the linking number of two pseudomanifolds $\Phi$ and $\Phi'$, $\Phi \subset R^n$, $\Phi' \subset R^n$, is equal to the linking number of the pseudomanifolds $C(\Phi)$ and $C(\Phi')$.*

*Proof.* Let $\mathfrak{z}^p$ be an orienting proper cycle of $\Phi$. Denote by $\mathfrak{z}_1{}^{q-1}$ a proper cycle in $\Gamma = R^n \setminus \Phi$ such that $\mathfrak{v}(\mathfrak{z}^p, \mathfrak{z}_1{}^{q-1}) = 1$ and such that $\mathfrak{z}_1{}^{q-1}$ is a $(J, \mathfrak{R})$-basis in $\Gamma$. An orienting cycle $\mathfrak{z}'^{q-1}$ of $\Phi'$ satisfies the relation

$$\mathfrak{z}'^{q-1} \sim a\mathfrak{z}_1{}^{q-1} \text{ in } \Gamma,$$

where $a = \mathfrak{v}(\mathfrak{z}^p, \mathfrak{z}'^{q-1})$ and $|a|$ is the linking number of $\Phi$ and $\Phi'$. Since a topological mapping of $R^n$ induces isomorphisms of the Betti groups of $\Gamma$ (see XIV, 3.3, where an even stronger theorem on the invariance of the Betti groups of a domain $\Gamma$ under topological mappings of $\Gamma$ onto itself is proved), the cycle $C(\mathfrak{z}'^{q-1})$ satisfies the relation $C(\mathfrak{z}'^{q-1}) \sim aC(\mathfrak{z}_1{}^{q-1})$ in $C(\Gamma)$, and the cycle $C(\mathfrak{z}_1{}^{q-1})$ forms a $(J, \mathfrak{R})$-basis of $C(\Gamma)$. If we choose in $C(\Phi)$ a cycle having a linking number equal to 1 with $C(\mathfrak{z}_1{}^{q-1})$, we see that this cycle is an orientation of the pseudomanifold $C(\Phi)$ (and not a multiple of the orientation); hence it follows that $C(\mathfrak{z}^p)$, which is also an orienting cycle of $C(\Phi)$, has a linking number equal to $\pm 1$ with $C(\mathfrak{z}_1{}^{q-1})$. Consequently,

$$\mathfrak{v}(C(\mathfrak{z}^p), C(\mathfrak{z}'^{q-1})) = a\mathfrak{v}(C(\mathfrak{z}^p), C(\mathfrak{z}_1{}^{q-1})) = \pm a,$$

i.e., the linking number of $C(\Phi)$ and $C(\Phi')$ is equal to $|a|$. This completes the proof.

### *Part Five*

# INTRODUCTION TO THE THEORY OF CONTINUOUS MAPPINGS OF POLYHEDRA

The theory of continuous mappings of polyhedra is one of the most important and interesting parts of topology. In particular, the theory of continuous mappings of manifolds, especially of spheres, formed the content of many notable topological investigations of recent decades. As in many other cases, the foundations of the theory were laid in the classical papers of Brouwer (see the Mathematische Annalen, vols. 70–72 (1911–1913)). The papers of Brouwer and Hopf, whose methods are related to those of Brouwer, furnish the fundamental content of the last two chapters of this book, particularly that of Chapter XVI. Chapter XVII is devoted to the remarkable fixed point theorem for polyhedra, proved by Lefschetz for manifolds and first extended by Hopf to arbitrary polyhedra.

The choice of material for the last two chapters was dictated not only by the finality, importance and classical nature of the results, but also by the esthetic appeal of the intuitive nature of the theorems and the methods of proof.

Finally, we must note the value of this type of question in the history of mathematics and particularly of topology. The value accrues for three reasons. First, topology, which is still a very young branch of mathematics, draws strength from the theory of continuous mappings of the simplest manifolds. Secondly, this theory yields the *first applications of topology to analysis* and this is evidence of the many bonds connecting topology to the rest of mathematics. Finally, this theory is responsible for the profound and continually increasing influence of topology on the development of the mathematical thought of our time.

As for the literature of the theory of continuous mappings of manifolds and polyhedra, in addition to the papers of Brouwer mentioned above, we cite the famous paper of Lefschetz on intersections in complexes and manifolds (see Lefschetz [a]). This paper was the first to give an algebraic theory of continuous mappings of manifolds based on intersections (the intersection number discussed in Chapter XIII is the simplest concept of this theory). Lefschetz's theory was further developed in Hopf's paper on the algebra of mappings of manifolds (see Hopf [b]; see also Hopf [d]). The same questions were treated by Hopf from the point of view of *homotopy* in his paper on the topology of mappings of manifolds (see Hopf [a]). This paper is recommended to the reader unreservedly because of the unusual quality of its expository style. For the special case of continuous mappings of surfaces, see Kneser [a] (this case, i.e., $n = 2$, proved to be insoluble by the general methods of Hopf).

Hopf's paper on the mappings of a 3-sphere into a 2-sphere (see Hopf [c]) began the research on the mappings of spheres of different dimensions into each other. The problem of the *classification of these mappings*, that is, the determination of the homotopy classes of the mappings of an $m$-sphere into an $n$-sphere, is trivial for $m < n$: in this case all the mappings of $S^m$ into $S^n$ are in one homotopy class, namely, the class containing the mapping which takes $S^m$ into one point of $S^n$. For $m = n$ the problem was solved by Hopf; its solution is given in XVI, 6. [The problem of classifying the continuous mappings of an $n$-dimensional polyhedron into an $n$-sphere was also solved by Hopf (see Alexandroff-Hopf, *Topologie*, *I*, Chapter XIII). The much more general problem of classifying the continuous mappings of an $n$-dimensional compactum into an $n$-sphere has also been solved by now. An excellent exposition of these questions can be found in Hurewicz-Wallman, *Dimension Theory*, Chapter VIII, which is highly recommended. Finally, the problem of classifying the mappings of an arbitrary $n$-dimensional normal space into an $n$-sphere has also been solved by essentially the same methods (see Aleksandrov [j (vol. 1, Bibliography)]).] However, the classification problem for $m > n$ has turned out to be extremely difficult. To this day the problem has been solved only in special cases. These solutions are due to the profound and difficult research of Hopf (see Hopf [c]), L. S. Pontryagin (see Pontryagin [b]) and others. Hopf's paper (mentioned above) on the mappings of a 3-sphere into a 2-sphere is an excellent introduction to this interesting and difficult part of topology.

Closely related to the work of Hopf on the theory of continuous mappings are, on the one hand, the papers of Freudenthal (see Freudenthal [a]) and Bruschlinsky (see Bruschlinsky [a]), and on the other, the very famous researches of Hurewicz on the homotopy groups (see Hurewicz [a]). A study of these papers should be preceded by an acquaintance with the fundamental group. For expositions of the latter, see Seifert-Threlfall, *Lehrbuch der Topologie*, Chapters 7 and 8; Pontryagin, *Topological Groups*, Chapter 8; or Hopf [a]. (For the books of Alexandroff-Hopf, Hurewicz-Wallman, Seifert-Threlfall and Pontryagin, see the Bibliography at the end of Volume 1 of this book).

Lately, cohomology theory has been applied to the theory of continuous mappings with great success. For an account of this see Chapter 8 of Hurewicz-Wallman and Pontryagin's paper (see Pontryagin [b]) on the classification of mappings of a 3-complex into a 2-sphere. In this paper Pontryagin makes the transition from the problem of classifying the mappings of a 3-sphere into a 2-sphere to the analogous problem for the mappings of a three-dimensional polyhedron into a 2-sphere. The paper also contains further references to the literature on these problems.

In view of its applications to analysis, an extremely important, but still hardly popular, field of topology is the investigation of the continuous mappings of infinite dimensional spaces, and especially function spaces. In this connection we note the papers of Leray-Schauder (see Leray [a]) and E. Rothe (see Rothe [a]).

Finally, we must cite the very interesting paper of J. Leray (see Leray [b]), devoted mainly to the duality theorems and continuous mappings.

[For a recent treatment of the classification of continuous mappings and further literature on the subject, see S. T. Hu, *Homotopy Theory*, Academic Press, New York and London, 1959.—Trans.]

## *Chapter XVI*

# THE BROUWER THEORY OF CONTINUOUS MAPPINGS IN $R^n$ AND $S^n$

In this chapter (which is a reworking of Chapter 12 of Alexandroff-Hopf's *Topologie, I*) we consider the elementary classical theorems on continuous mappings, in essence due to Brouwer, and also some newer results closely related to these theorems; the most important of the latter is Hopf's classification of the continuous mappings of one $n$-sphere into another. The proofs given here are due to Hopf; they originated in the methods of Brouwer and represent a further development and refinement of these methods. The proofs are based on the concept of the degree of a mapping (VIII, 5.1 and XI, 6) and on the systematic application of a simple and geometrically quite intuitive case of linking of a pair of cycles $z^r$, $z^s$ of $R^n$ (see the introductory remarks at the beginning of XV; we shall use the same notation in this chapter): the case $r = 0$, $z^r$ a point $o \in R^n$ with coefficient 1, and $z^s$ an $(n - 1)$-cycle whose carrier does not contain $o$.

In the sequel, mappings of point sets are always continuous mappings.

## §1. Index of a point relative to an $(n - 1)$-cycle in $R^n$

**§1.1. The combinatorial case.** We consider the linking number $\mathfrak{v}(z^{n-1}, o)$, where $o$ is a point of (an oriented) $R^n$ taken with coefficient 1 and $z^{n-1}$ is an $(n - 1)$-cycle such that $o \notin \bar{z}^{n-1}$ (if $n = 1$, we assume in addition that $z^{n-1}$ is a normal cycle). The linking number $\mathfrak{v}(z^{n-1}, o)$ is called *the index of the point $o$ relative to the cycle $z^{n-1}$*. We note that if $o'$ is a point exterior to a solid sphere $E^n$ containing the body $\bar{z}^{n-1}$ of the chain $z^{n-1}$ in its interior (see XV, 0.1; the results of XV, 1, 2 are presupposed in this chapter), then, setting $z^0 = o - o'$, we obviously have

$$\mathfrak{v}(z^{n-1}, o) = \mathfrak{v}(z^{n-1}, z^0). \tag{1.1}$$

In the sequel, it will be convenient to assume that the point $o'$ is chosen to satisfy the following condition: the segment $oo'$ is in general position relative to the complex $| z^{n-1} |$ [i.e., it may intersect only nondegenerate $(n - 1)$-simplexes of $| z^{n-1} |$].

The index of a point $o$ relative to a cycle $z^{n-1}$, i.e., $\mathfrak{v}(z^{n-1}, o)$, can be interpreted in two ways:

*First interpretation.*

$$\mathfrak{v}(z^{n-1}, o) = (x^n \times o), \tag{1.11}$$

where $x^n$ is an arbitrary chain bounded by $z^{n-1}$ which is in general position with respect to $o$. (The latter condition means, in this chapter, that the

$n$-simplexes of the complex $|x^n|$ are nondegenerate and that $o$ can belong only to such simplexes.)

If $x^n = \sum c_i t_i^n$ and $o \in |t_i^n|$, then $(t_i^n \times o) = t_i^n R^n$ (see VII, 2.1). If the simplexes $|t_i^n| \in |x^n|$ which contain $o$ are $|t_1^n|, \cdots, |t_s^n|$, then

$$\mathfrak{v}(z^{n-1}, o) = \textstyle\sum_{i=1}^{s} c_i(t_i^n R^n). \tag{1.12}$$

If the coefficients $c_i$ of the simplexes $t_i^n$ in $x^n$ for which $i \leq s$ and $t_i^n R^n = 1$ are denoted by $c_i^+$ and those for which $t_j^n R^n = -1$ are denoted by $c_j^-$, then (1.12) may be rewritten as

$$\mathfrak{v}(z^{n-1}, o) = \sum c_i^+ - \sum c_j^-. \tag{1.12′}$$

The right side of (1.12) or (1.12′) is called the *multiplicity* of the point $o$ with respect to the chain $x^n$ (it is the algebraic number of times the chain $x^n$ "covers" the point $o$). Hence, in view of (1.12′), *the index* $\mathfrak{v}(z^{n-1}, o)$ *may be defined as the number of times the point $o$ is covered by an arbitrary chain $x^n$ with boundary $z^{n-1}$ and in general position relative to $o$.*

*Second interpretation.* This is obtained by comparing (1.1) with XV, (1.4) (retaining the conditions on $o'$ and the cycle $z^0 = o - o'$ given above). XV, (1.4) yields (for $p = 0$, $q = n - 1$)

$$\mathfrak{v}(z^0, z^{n-1}) = -\mathfrak{v}(z^{n-1}, z^0). \tag{1.13}$$

Choose the orientation $|oo'|$ of the segment $(oo')$. Since

$$\Delta|oo'| = o' - o = -z^0,$$

(1.13) gives

$$(|oo'| \times z^{n-1}) = -\mathfrak{v}(z^0, z^{n-1}) = \mathfrak{v}(z^{n-1}, z^0) = \mathfrak{v}(z^{n-1}, o). \tag{1.14}$$

Now let $(o, \infty)$ be the ray with initial point $o$ and passing through $o'$. It is clear that the intersection number $(|oo'| \times z^{n-1})$ is constant for all $o'$ on the ray and sufficiently far from $o$. We denote this constant value by $[(o, \infty) \times z^{n-1}]$.

Formula (1.14) can therefore be replaced by

$$[(o, \infty) \times z^{n-1}] = \mathfrak{v}(z^{n-1}, o). \tag{1.15}$$

Hence,

*The index of a point $o$ relative to a cycle*

$$z^{n-1} \in Z^{n-1}(R^n \setminus o, \mathfrak{A})$$

*is the intersection number $[(o, \infty) \times z^{n-1}]$ of an arbitrary ray, with initial point $o$ and in general position relative to $z^{n-1}$, and the cycle $z^{n-1}$.*

This is the second interpretation of the index of a point.

**§1.2. The index of a point $o \in R^n$ relative to a proper cycle $\mathfrak{z}^{n-1}$.** Without

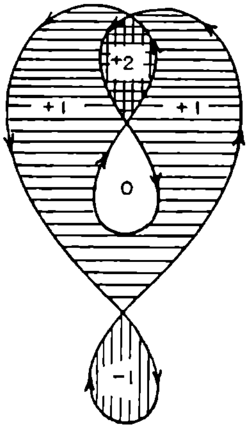


FIG. 161

loss of generality, the reader may restrict himself in this chapter to proper cycles of the following types:

1°) $$\mathfrak{z}_\alpha^r = (z_\alpha^r, z_{\alpha 1}^r, \cdots, z_{\alpha h}^r, \cdots),$$

where $z_\alpha^r$ is a cycle of a triangulation $K_\alpha$ and $z_{\alpha h}^r$ is the barycentric subdivision of order $h$ of $z_\alpha^r$; 2°) $\mathfrak{z}^r = C\mathfrak{z}_\alpha^r$, i.e., the image of a proper cycle of type 1°) under a continuous mapping $C$ of the polyhedron $\| K_\alpha \|$ into $R^n$ (or $S^n$).

Let $\mathfrak{z}^{n-1}$ be a proper cycle of a compactum $\Phi \subset R^n$ and suppose that $o \in \Gamma = R^n \setminus \Phi$; then the linking number $\mathfrak{v}(\mathfrak{z}^{n-1}, o)$ is called *the index of the point o relative to the proper cycle* $\mathfrak{z}^{n-1}$. In view of XV, Theorem 1.55 it follows that:

If $o, o' \in \Gamma$ and

(1.21) $$o \sim o' \text{ in } \Gamma,$$

then

$$\mathfrak{v}(\mathfrak{z}^{n-1}, o) = \mathfrak{v}(\mathfrak{z}^{n-1}, o').$$

But the homology (1.21) means that $o$ and $o'$ can be joined by a polygonal line in $\Gamma$, i.e., that they are contained in the same component of $\Gamma$. Hence,

1.2. *If a compactum* $\Phi \subset R^n$ *is a carrier of a proper cycle* $\mathfrak{z}^{n-1}$, *then the index* $\mathfrak{v}(\mathfrak{z}^{n-1}, o)$ *relative to the cycle* $\mathfrak{z}^{n-1}$ *is defined for all* $o \in \Gamma$, *and* $\mathfrak{v}(\mathfrak{z}^{n-1}, o)$ *assumes the same value at all points o contained in the same component of* $\Gamma$.

In Fig. 161 the curve $\Phi$ is the carrier of the singular cycle $\mathfrak{z}^1$ (here $n = 2$). The various values (0, 2, 1, −1) assumed by $\mathfrak{v}(\mathfrak{z}^1, o)$ (depending on the component of $\Gamma = R^2 \setminus \Phi$ containing $o$) are indicated in the figure.

Remark 1. In view of the discussion in 1.1, we may interpret $\mathfrak{v}(\mathfrak{z}^{n-1}, o)$ as follows:

*First interpretation.* Choose

$$\mathfrak{z}^{n-1} = (z_1^{n-1}, z_2^{n-1}, \cdots, z_k^{n-1}, \cdots)$$

and the chains $x_k^n$ bounded by the cycles $z_k^{n-1}$ so that they are in general position relative to $o$; then the intersection number $(x_k^n \times o)$ is constant and equal to $\mathfrak{v}(\mathfrak{z}^{n-1}, o)$ for all sufficiently large $k$.

*Second interpretation.* If $k$ is sufficiently large, then $\mathfrak{v}(z^{n-1}, o) = [(o, \infty) \times z_k^{n-1}]$, where $(o, \infty)$ is any ray with initial point $o$ in general position with respect to $z_k^{n-1}$.

Remark 2. If $\mathfrak{z}^{n-1} \sim 0$ in $\Phi$, then $\mathfrak{v}(z^{n-1}, o) = 0$ for all $o \in \Gamma$. It follows that *if* $\mathfrak{z}^{n-1} \sim \mathfrak{z}'^{n-1}$ *in* $\Phi$, *then* $\mathfrak{v}(\mathfrak{z}^{n-1}, o) = \mathfrak{v}(\mathfrak{z}'^{n-1}, o)$ *for all* $o \in \Gamma$.

Remark 3. If $T^n$ is an $n$-simplex containing $o$ and $t^n$ is an orientation of $T^n$, then $\mathfrak{v}(\Delta t^n, o) = \pm 1$. Hence $\Delta t^n \nsim 0$ in $R^n \setminus o$.

**§1.3. The Poincaré-Bohl theorem and Rouché's theorem.** We note a special case of Remark 2.

1.3. *If two continuous mappings* $C_0$ *and* $C_1$ *of a compactum* $\Phi$ *into* $R^n \setminus o$ *are homotopic in* $R^n \setminus o$ *and* $\mathfrak{z}^{n-1}$ *is a proper cycle of* $\Phi$, *then* $\mathfrak{v}(C_0\mathfrak{z}^{n-1}, o) = \mathfrak{v}(C_1\mathfrak{z}^{n-1}, o)$.

*Proof.* If $C_0$ and $C_1$ are homotopic in $R^n \setminus o$, they are also homotopic in a compactum $\Phi_0 \subset R^n \setminus o$ (as $\Phi_0$ we may take, for instance, a sufficiently large closed cube from which is deleted a sufficiently small neighborhood of $o$). Therefore $C_0\mathfrak{z}^{n-1} \sim C_1\mathfrak{z}^{n-1}$ in $\Phi_0 \subset R^n \setminus o$, so that $\mathfrak{v}(C_0\mathfrak{z}^{n-1}, o) = \mathfrak{v}(C_1\mathfrak{z}^{n-1}, o)$.

The hypothesis of Theorem 1.3 is satisfied, in particular, if the segment $C_0(p)C_1(p) \subset R^n \setminus o$ for every $p \in \Phi$ [it is sufficient to define $C_\theta(p)$ as the point dividing the segment $C_0(p)C_1(p)$ in the ratio $\theta:(1 - \theta)$ to obtain a deformation $C_\theta$, $0 \leq \theta \leq 1$, which takes $C_0$ into $C_1$). This special case of Theorem 1.3 is known as the Poincaré-Bohl theorem.

Still another special case of Theorem 1.3 is obtained by requiring that the mappings $C_0$ and $C_1$ satisfy the condition: $\rho[C_0(p), C_1(p)] < \rho[C_0(p), o]$ for all $p \in \Phi$. This special case of Theorem 1.3 (and the Poincaré-Bohl theorem) is sometimes called Rouché's theorem.

We note, finally, that the hypotheses of Theorem 1.3 (and hence of the Poincaré-Bohl theorem) are satisfied if both $C_0$ and $C_1$ are mappings of a compactum $\Phi$ into a sphere $S^{n-1}$ with center $o$ such that $C_0(p)$ *and* $C_1(p)$ *are not diametrically opposite points of* $S^{n-1}$ *for any* $p \in \Phi$. We shall use this remark in §5.

**§1.4. Homologies in $R^n \setminus o$.** We assume that $R^n$ is given a fixed orientation and that $t^n$ is an arbitrary oriented simplex of $R^n$ whose orientation is coherent with that of $R^n$. Denote by $o$ any point of $T^n = |t^n|$; then, in view of (1.11), $\mathfrak{v}(\Delta t^n, o) = 1$.

THEOREM 1.41. *Every proper cycle* $\mathfrak{z}^{n-1}$ *of an arbitrary compactum* $\Phi \subset R^n \setminus o$ *satisfies*

$$\mathfrak{z}^{n-1} \sim \mathfrak{v}(\mathfrak{z}^{n-1}, o)\Delta t^n \quad \textit{in} \quad R^n \setminus o, \tag{1.41}$$

*where* $o \in |t^n|$.

*Proof.* Let $C_0$ be the identity mapping of $\Phi$, and let $C_1$ be the projection of $\Phi$ from $o$ into $\bar{T}^n \setminus T^n$, i.e., the mapping which assigns to every $p \in \Phi$ the point of intersection of the ray $(op)$ with $\bar{T}^n \setminus T^n$.

The mappings $C_0$ and $C_1$ satisfy the hypotheses of the Poincaré-Bohl theorem and are therefore homotopic, so that

$$\mathfrak{z}^{n-1} \sim C_1\mathfrak{z}^{n-1} \quad \text{in} \quad R^n \setminus o.$$

Since the proper cycle $C_1\mathfrak{z}^{n-1}$ is contained in the boundary of $T^n$, it follows that $C_1\mathfrak{z}^{n-1} \sim \gamma\Delta t^n$ in the boundary of $T^n$ and hence in $R^n \setminus o$. Therefore, $\mathfrak{z}^{n-1} \sim \gamma\Delta t^n$ in $R^n \setminus o$, and so

$$\mathfrak{v}(\mathfrak{z}^{n-1}, o) = \mathfrak{v}(\gamma\Delta t^n, o) = \gamma\mathfrak{v}(\Delta t^n, o) = \gamma.$$

This completes the proof.

REMARK. Since $\Delta t^n \not\sim 0$ in $R^n \setminus o$, it follows that $\mathfrak{z}^{n-1} \sim 0$ in $R^n \setminus o$, in view of 1.41, implies that $\mathfrak{v}(\mathfrak{z}^{n-1}, o) = 0$.

Another consequence of 1.41 is the following proposition:

If $\mathfrak{v}(\mathfrak{z}^{n-1}, o) = 0$, then $\mathfrak{z}^{n-1} \sim 0$ in $R^n \setminus o$. In other words, in order that $\mathfrak{z}^{n-1} \sim 0$ in $R^n \setminus o$ it is not only *necessary* but also *sufficient* that $\mathfrak{v}(\mathfrak{z}^{n-1}, o) = 0$.

Hence,

THEOREM 1.42. *Two proper cycles* $\mathfrak{z}_1^{n-1}$ *and* $\mathfrak{z}_2^{n-1}$, *neither of which contains* $o$, *are homologous in* $R^n \setminus o$ *if, and only if,* $\mathfrak{v}(\mathfrak{z}_1^{n-1}, o) = \mathfrak{v}(\mathfrak{z}_2^{n-1}, o)$.

**§1.5. The interpretation of the index of a point relative to a cycle as the degree of the central projection of the cycle onto a sphere.** We again assume that $o \in T^n$ and that $t^n$ is the orientation of $T^n$ coherent with the chosen orientation of $R^n$. We refer to the orientation $\Delta t^n$ of the boundary of $T^n$ as the orientation *induced* by that of $R^n$.

REMARK. If $S^{n-1}$ is a sphere with center $o$, we define the orientation induced on it by the prescribed orientation of $R^n$ as that orientation of $S^{n-1}$ into which the induced orientation $\Delta t^n$ of the pseudomanifold $\bar{T}^n \setminus T^n$ is mapped by a central projection with center $o$.

We again denote by $C_1$ the projection of $\Phi \subset R^n \setminus o$ from $o$ into the boundary of $T^n$, and by $\mathfrak{z}^{n-1}$ a proper cycle on $\Phi$. Since $\Delta t^n$ is the orientation of the pseudomanifold $S^{n-1}$,

$$C_1\mathfrak{z}^{n-1} \sim \gamma\Delta t^n \quad \text{in} \quad S^{n-1}. \tag{1.5}$$

In view of the Remark following Theorem 1.41, it immediately follows that $\gamma = \mathfrak{v}(\mathfrak{z}^{n-1}, o)$. But according to the definition of the degree of a mapping, the number $\gamma$ is the degree of the mapping $C_1$ of the proper cycle $\mathfrak{z}^{n-1}$ into the oriented pseudomanifold $\bar{T}^n \setminus T^n$ (an orienting cycle is $\Delta t^n$). Hence,

THEOREM 1.5. *The index of a point $o$ relative to a proper cycle $\mathfrak{z}^{n-1}$ contained in $R^n \setminus o$ is equal to the degree of the projection of this cycle from $o$ into the boundary of a simplex $T^n$ containing $o$, where the boundary has the orientation induced by $R^n$.*

REMARK 2. If $\mathfrak{z}^{n-1}$ is a singular cycle

$$\mathfrak{z}^{n-1} = C\mathfrak{z}_\alpha^{n-1}, \tag{1.51}$$

then the degree of the mapping $C_1$ of $\mathfrak{z}^{n-1}$ is obviously equal to the degree of the mapping $C_1C$ of the cycle $\mathfrak{z}_\alpha^{n-1}$. Therefore,

1.51. The index of a point $o$ relative to a singular cycle $\mathfrak{z}^{n-1}$ contained in $R^n \setminus o$, where $\mathfrak{z}^{n-1}$ has the parametric representation (1.51), is equal to the degree of the mapping $C_1C$ of the cycle $\mathfrak{z}_\alpha^{n-1}$ into the boundary of a simplex $T^n$ containing $o$, where the boundary has the orientation induced by $R^n$. In particular, if the cycle $\mathfrak{z}^{n-1}$ is contained in $S^{n-1} = \bar{T}^n \setminus T^n$, i.e., if $C$ is a mapping into $S^{n-1}$, then $\mathfrak{v}(\mathfrak{z}^{n-1}, o)$ coincides with the degree of the mapping $C$ of the cycle $\mathfrak{z}_\alpha^{n-1}$ into $S^{n-1}$.

EXERCISE. Let $C$ be a continuous mapping of an oriented circumference $z^1$ into $R^2 \setminus o$. Prove that the index of the point $o$ relative to $Cz^1$ coincides with the index defined in II, 2.2.

## §2. The existence of roots

**§2.1. Existence theorem for roots.** Let $\mathfrak{z}$ be a proper $(n-1)$-cycle in $R^n$ with carrier $\Phi$:

$$\mathfrak{z} = (z_1, z_2, \cdots, z_n, \cdots),$$

and let $o$ be a point of the open set $\Gamma = R^n \setminus \Phi$. We know from the preceding chapter that if $\mathfrak{z} \sim 0$ in $R^n \setminus o$, then the index $\mathfrak{v}(\mathfrak{z}, o) = 0$. It now immediately follows that:

THEOREM 2.1. (GENERAL THEOREM ON THE EXISTENCE OF ROOTS.) *Let $\mathfrak{z}_0^{n-1}$ be a proper cycle with carrier $\Psi_0$, homologous to zero in a compactum $\Phi_0 \supset \Psi_0$.*

*Let $C$ be a continuous mapping of $\Phi_0$ into $R^n$ and suppose that*

$$o \in R^n \setminus C(\Psi_0).$$

*If*

$$\mathfrak{v}(C\mathfrak{z}_0^{n-1}, o) \neq 0,$$

*there exists at least one point* $p_0 \in \Phi_0$ *such that* $Cp_0 = o$. (The point $p_0$ is called a root of $C$.)

Indeed,

$$C\mathfrak{z}_0^{n-1} \sim 0 \quad \text{in} \quad C\Phi_0\,;$$

hence if

$$C\Phi_0 \subset R^n \setminus o,$$

then

$$C\mathfrak{z}_0^{n-1} \sim 0 \quad \text{in} \quad R^n \setminus o,$$

so that

$$\mathfrak{v}(C\mathfrak{z}_0^{n-1}, o) = 0,$$

in contradiction to our assumptions.

In most applications of this theorem $\mathfrak{z}_0^{n-1}$ is a cycle of a triangulation $K_0$, homologous to zero in $K_0$. The following special case is particularly important:

2.10. *Let* $\bar{E}^n$ *be a solid n-sphere,* $S^{n-1}$ *its bounding* $(n - 1)$*-sphere,* $\mathfrak{z}^{n-1}$ *a definite orientation of* $S^{n-1}$, $C$ *a continuous mapping of* $\bar{E}^n$ *into* $R^n$ *and* $o \in R^n \setminus CS^{n-1}$. *If* $\mathfrak{v}(C\mathfrak{z}^{n-1}, o) \neq 0$, *there exists at least one point* $p \in \bar{E}^n$ *such that* $Cp = o$.

REMARK. A second special case of Theorem 2.1 has to do with polyhedral domains in $R^n$.

In this chapter we shall understand a *polyhedral domain* in $R^n$ to mean a domain of the form $\Gamma = T^n \setminus (\bar{T}_1^n \cup \cdots \cup \bar{T}_\nu^n)$, where the closed $n$-simplexes $\bar{T}_1^n, \cdots, \bar{T}_\nu^n$ are contained in the simplex $T^n$ and are *pairwise disjoint.*

Let $K_\alpha^n$ be a triangulation of a polyhedron $\bar{\Gamma}$. We impose the orientations $t_{\alpha i}^n$, which are coherent with a prescribed orientation $t^n$ of $T^n$, on all the simplexes $T_{\alpha i}^n$ of $K_\alpha^n$.

The cycle $\sum \Delta t_{\alpha i}^n = z_\alpha^{n-1}$ is called the *boundary of the oriented domain* $\Gamma$ (in the triangulation $K_\alpha^n$). Under these conditions, the following special case of Theorem 2.1 holds:

2.11. *Let* $\Gamma$ *be a polyhedral domain in* $R^n$, *and let* $C$ *be a continuous mapping of* $\Gamma$ *into a space* $R_1^n$. *Let* $o$ *be a point of* $R_1^n$ *not contained in* $C(\bar{z}_\alpha^{n-1})$, *where* $z_\alpha^{n-1}$ *is the boundary of the polyhedral domain* $\Gamma$ (see above). *If* $\mathfrak{v}(Cz_\alpha^{n-1}, o) \neq 0$, *there exists at least one point* $p \in \Gamma$ *such that* $C(p) = o$.

## §3. The local degree of a mapping of an $n$-chain into $R^n$

INTRODUCTORY REMARK. If $x$ is a chain of a triangulation, we shall understand a continuous mapping of the chain $x$ to mean a continuous mapping of its body, the polyhedron $\bar{x}$.

**§3.1. Definition and fundamental properties of the local degree.** Let $C$ be a continuous mapping of an $n$-chain into an oriented space $R^n$ and suppose that $o \in R^n$. We assume that $C$ has no roots on $\bar{\Delta}x^n$, i.e., that no point of the polyhedron $\bar{\Delta}x^n$ is mapped into $o$. Under these conditions, the number $\mathfrak{v}(C\Delta x^n, o) = \gamma(o, x^n, C)$ is called the *degree of $C$ in the point $o$*.

Remark. We shall sometimes omit some of the arguments in $\gamma(o, x^n, C)$.

The following are consequences of the above definition and the elementary properties of the index of a point relative to a cycle:

3.11. If $x^n = x_1^n + x_2^n$ and no point of $\bar{\Delta}x_1^n \cup \bar{\Delta}x_2^n$ is mapped into $o$, then

$$\gamma(x^n) = \gamma(x_1^n) + \gamma(x_2^n).$$

3.12. We have

$$\gamma(ax^n) = a\gamma(x^n),$$

where $a$ is an arbitrary integer.

Hence,

3.13. If $x^n = \sum a_i t_i^n$ and every point of $C^{-1}(o)$ is contained in some $| t_i^n |$ (so that $C\bar{\Delta}t_i^n \subset R^n \setminus o$ for arbitrary $i$), then

$$\gamma(o, x^n, C) = \sum a_i\gamma(o, t_i^n, C).$$

Furthermore,

3.14. If $C_0$ and $C_1$ are continuous mappings of a chain $x^n$ into $R^n$ and $C_\theta$ is a deformation of $C_0$ (restricted to $\bar{\Delta}x^n$) into $C_1$ such that $C_\theta(\bar{\Delta}x^n) \subset R^n \setminus o$ for all $\theta$, $0 \leq \theta \leq 1$, then $\gamma(o, x^n, C_0) = \gamma(o, x^n, C_1)$.

In particular (the theorems of Poincaré-Bohl and Rouché),

3.140. If $o$ is not contained in the segment $C_0(p)C_1(p)$ for any $p \in \bar{\Delta}x^n$, then

$$\gamma(o, x^n, C_0) = \gamma(o, x^n, C_1).$$

3.141. If

$$\rho(C_0p, C_1p) < \rho(C_0p, o)$$

for all $p \in \bar{\Delta}x^n$, then

$$\gamma(o, x^n, C_0) = \gamma(o, x^n, C_1).$$

Finally, we have the very important corollary

3.15. *If $\gamma(o, x^n, C)$ is defined, then the degree $\gamma(o, x^n, C_1)$ is defined and equal to $\gamma(o, x^n, C)$ for every continuous mapping $C_1$ of the chain $x^n$ into $R^n$, which is sufficiently near $C$ on $\bar{\Delta}x^n$.*

**§3.2. Simplicial approximations.** We now let $C_1$ be an arbitrary sufficiently good simplicial approximation to $C$ (see XI, 5.1) satisfying the following condition: the vertices of the subdivision $| x_1^n |$ of the complex $| x^n |$ defining the mapping $C_1$ are chosen so that the point $o$ is not contained in any of the sets $C(T^r)$, where $T^r \in |x_1^n|$ and $r < n$. If $s_1$ is the subdivision operator, then

$$\Delta s_1 x^n = s_1 \Delta x^n \sim \Delta x^n \quad \text{on } \bar{\Delta} x^n,$$

so that

$$C_1 \Delta s_1 x^n \sim C_1 \Delta x^n \quad \text{on } \bar{\Delta} x^n \subset R^n \setminus o,$$

i.e.,

$$\mathfrak{v}(C_1 \Delta s_1 x^n, o) = \mathfrak{v}(C_1 \Delta x^n, o),$$

$$\gamma(o, s_1 x^n, C_1) = \gamma(o, x^n, C_1).$$

But, in view of 3.15, $\gamma(o, x^n, C) = \gamma(o, x^n, C_1)$, so that

$$\gamma(o, x^n, C) = \gamma(o, s_1 x^n, C_1), \tag{3.21}$$

that is,

3.21. *The degree of a continuous mapping $C$ of a chain $x^n$ in the point $o \in R^n \setminus C(\bar{\Delta} x^n)$ is equal to the degree of an arbitrary simplicial mapping (of a subdivision $s_1 x^n$ of $x^n$), provided it is a sufficiently good approximation to $C$.*

We may also always require that $C_1$ satisfy the following *regularity condition: All the roots of $C_1$ [that is, all the points of $C^{-1}(o)$] are contained in the n-simplexes of the complex* $|s_1 x^n|$ (where each of these simplexes contains no more than one root).

Since Theorem 3.13 is applicable to a regular mapping, the calculation of the local degree of an arbitrary continuous mapping is reduced to the computation of the number $\gamma(o, t^n, C)$, where $C$ is a nondegenerate affine mapping of $t^n$ into $R^n$, with $o \in C|t^n|$. But then, setting $t'^n = Ct^n$, we have

$$\gamma(o, t^n, C) = t'^n R^n,$$

i.e., $\gamma = 1$ if the orientations $t'^n$ and $R^n$ are coherent and $-1$ if they are not. This is clear because in the first case the intersection number $(t'^n \times o)$ is equal to 1, whereas in the second case it is $-1$.

Combining this result with Theorem 3.13, we obtain

3.22. If $C$ is a regular simplicial mapping of a chain $x^n = \sum a_i t_i^n$ into $R^n$, then

$$\gamma(o, x^n, C) = \sum a_i \gamma_i,$$

where 1) $\gamma_i = 0$ if $o \notin C|t_i^n|$, 2) $\gamma_i = 1$ if $o \in C|t_i^n|$ and the orientations $Ct_i^n$ and $R^n$ are coherent and 3) $\gamma_i = -1$ if $o \in C|t_i^n|$ and the orientations $Ct_i^n$ and $R^n$ are not coherent.

In particular, if $x^n = \sum t_i^n$ is an oriented combinatorial pseudomanifold, then

$$\gamma(o, x^n, C) = \sum \gamma_i = \pi - \nu,$$

where $\pi$ is the number of simplexes $t_i^n$ whose images under $C$ cover $o$ "positively" and $\nu$ is the number of simplexes $t_i^n$ whose images under $C$ cover $o$ "negatively".

**§3.3. Mappings into $S^n$.**

THEOREM 3.3 *Suppose the spheres $S_0^n$ and $S^n$ to be oriented and let $C$ be a continuous mapping of $S_0^n$ into $S^n$ of degree $\gamma_C$. We think of the sphere $S^n$ as the Euclidean space $R^n$ compactified by the addition of a single point at infinity $u$:*

$$S^n = R^n \cup u.$$

*We assume that*

$$C^{-1}(o) \subset T_0^n$$

*for some $o \in R^n$, where $T_0^n$ is an $n$-simplex of a triangulation $K_0$ of $S_0^n$, and denote by $t_0^n$ the orientation of $T_0^n$ coherent with a prescribed orientation of $S_0^n$. Then*

$$\mathfrak{v}(C\Delta t_0^n, o) = \gamma_C. \tag{3.3}$$

*Proof.* Both sides of (3.3) remain unchanged if $C$ is replaced by a sufficiently good simplicial approximation. We may therefore begin by assuming that $C$ is a simplicial mapping of a subdivision $K_\alpha$ of $K_0$ into a triangulation $K$ of $S^n$, with $o$ contained in an $n$-simplex $T^n \in K$. We denote the orientation of $T^n$ coherent with the prescribed orientation of $S^n$ by $t^n$. We further assume that the point $u$ of $S^n$ (the point at infinity of $R^n$) is not contained in $\bar{T}^n$.

Let $t_{\alpha i}^n$ be the simplexes of $K_\alpha$ obtained from the subdivision of $T_0^n$ and oriented coherently with $S_0^n$. Finally, let $\pi$ and $\nu$ be, respectively, the number of $t_{\alpha i}^n$ for which $Ct_{\alpha i}^n = t^n$ and $Ct_{\alpha i}^n = -t^n$. Then, in view of VIII, Theorem 5.21, $\gamma_C = \pi - \nu$. On the other hand, if $x_0^n = \sum_i t_{\alpha i}^n$,

$$\mathfrak{v}(C\Delta t_0^n, o) = \mathfrak{v}(C\Delta x_0^n, o) = (Cx_0^n \times o) = \pi - \nu.$$

This completes the proof.

## §4. Topological mappings

**§4.1. Theorem 4.1.** *Suppose that $T^n$ is an $n$-simplex, $p \in T^n$ and $C$ is a topological mapping of $T^n$ into $R^n$ such that $Cp = o$. If $t^n$ is an arbitrary orientation of $T^n$, then*

$$\gamma(o, t^n, C) = \pm 1.$$

*Proof.* Let $S^{n-1}$ be the boundary of $T^n$. Since $\Phi = CS^{n-1}$ is a closed $(n-1)$-dimensional pseudomanifold and $C\Delta t^n$ is an orientation of $CS^{n-1}$, it follows from the fundamental Theorem 3.43 of Chapter XV that

$$\mathfrak{v}(C\Delta t^n, o') = \pm 1$$

for all points $o'$ in one of the two components into which $CS^{n-1}$ divides $R^n$. According to Theorem 2.1, there must exist a point $p' \in \bar{t}^n$ such that $Cp' = o'$. Because $C$ is $(1-1)$, $p'$ is contained in $\bar{t}^n - \bar{\Delta}t^n$, i.e., $p'$ is in $T^n$. The image of the segment $pp'$ under $C$ is a continuous path in $R^n \setminus C\bar{\Delta}t^n$, so that $o$ and $o'$ are in the same component of $R^n \setminus C\bar{\Delta}t^n$. Hence

$$\gamma(o, t^n, C) = \mathfrak{v}(C\Delta t^n, o) = \mathfrak{v}(C\Delta t^n, o') = \pm 1.$$

The following proposition is a stronger form of Theorem 4.1:

THEOREM 4.11. *Let* $\Gamma$ *be a domain of an oriented* $R^n$ *and let* $C$ *be a topological mapping of* $\Gamma$ *into an oriented* $R'^n$ *(both* $R^n$ *and* $R'^n$ *are Euclidean* $n$*-spaces). Suppose that* $|t_1^n|$ *and* $|t_2^n|$ *are arbitrary simplexes, oriented coherently with* $R^n$*, such that* $\bar{t}_1^n, \bar{t}_2^n \subset \Gamma$.

*If* $o_1 \in C|t_1^n|$, $o_2 \in C|t_2^n|$, *then*

$$\gamma(o_1, t_1^n, C) = \gamma(o_2, t_2^n, C) = \pm 1.$$

*Proof.* First, suppose that $|t_1^n| \cap |t_2^n| \neq 0$, and let $|t_3^n|$ be a simplex oriented coherently with $R^n$ such that $\bar{t}_3^n \subset |t_1^n| \cap |t_2^n|$ and suppose $o \in |t_3^n|$. As in the proof of 4.1, we conclude that

$$\gamma(o_1, t_1^n, C) = \gamma(o, t_1^n, C).$$

But

$$\gamma(o, t_1^n, C) = \gamma(o, t_3^n, C) = \gamma(o, t_2^n, C) = \gamma(o_2, t_2^n, C),$$

which proves the theorem in this special case.

Since any two simplexes $|t_1^n|$ and $|t_2^n|$ in $\Gamma$ can be joined by a chain of $n$-simplexes of $\Gamma$, so that two successive simplexes of the chain intersect, the theorem is also true in the general case.

If

$$\gamma(o, t^n, C) = 1$$

for arbitrary $\bar{t}^n \subset \Gamma$, $o \in C|t^n|$, we say that *the topological mapping* $C$ *of the domain* $\Gamma$ *into* $R^n$ *has degree* 1; in the contrary case [i.e., if $\gamma(o, t^n, C) = -1$ for all $\bar{t}^n \subset \Gamma$, $o \in C|t^n|$] we say that $C$ *has degree* $-1$.

## §4.2. Invariance theorems.

THEOREM 4.21. (INVARIANCE OF THE INDEX $\mathfrak{v}$.) *Suppose that* $o_1 \in R_1^n$ *and that* $\mathfrak{z}_1$ *is a proper* $(n-1)$*-cycle in* $R_1^n \setminus o_1$. *If* $|t_1^n|$ *is an* $n$*-simplex of* $R_1^n$ *containing both* $o_1$ *and* $\bar{\mathfrak{z}}_1$, $C$ *a topological mapping of* $\bar{t}_1^n$ *into* $R_2^n$ *and*

$\tau = \pm 1$ *the degree of* $C$, *then*

$$\mathfrak{v}(C\mathfrak{z}_1, Co_1) = \tau\mathfrak{v}(\mathfrak{z}_1, o_1).$$

*Proof.* Let $|t_2^n|$ be an $n$-simplex of $R_2^n$ containing $o_2 = Co_1$, and suppose that $|t_1^n|$, $|t_2^n|$ are oriented coherently with their carrying spaces $R_1^n$ and $R_2^n$. We denote $C\mathfrak{z}_1$ by $\mathfrak{z}_2$.

According to Theorem 1.41,

$$\mathfrak{z}_1 \sim \mathfrak{v}(\mathfrak{z}_1, o_1)\Delta t_1^n \quad \text{in } \bar{t}_1^n \setminus o_1, \tag{4.21}$$

$$\mathfrak{z}_2 \sim \mathfrak{v}(\mathfrak{z}_2, o_2)\Delta t_2^n \quad \text{in } R_2^n \setminus o_2. \tag{4.22}$$

In view of the definition of the degree $\tau$, we have

$$\mathfrak{v}(C\Delta t_1^n, o_2) = \tau;$$

consequently, by Theorem 1.41,

$$C\Delta t_1^n \sim \tau\Delta t_2^n \quad \text{in } R_2^n \setminus o_2. \tag{4.23}$$

Since $C(|t_1^n| \setminus o_1) \subset R_2^n \setminus o_2$, (4.21) yields

$$C\mathfrak{z}_1 = \mathfrak{z}_2 \sim \mathfrak{v}(\mathfrak{z}_1, o_1)C\Delta t_1^n \quad \text{in } R_2^n \setminus o_2;$$

that is, if (4.23) is taken into account,

$$\mathfrak{z}_2 \sim \tau\mathfrak{v}(\mathfrak{z}_1, o_1)\Delta t_2^n \quad \text{in } R_2^n \setminus o_2. \tag{4.24}$$

In view of (4.22), (4.24) implies that $\mathfrak{v}(\mathfrak{z}_2, o_2) = \tau\mathfrak{v}(\mathfrak{z}_1, o_1)$, which was to be proved.

Theorem 4.21 and the definition of the local degree of a mapping now imply

THEOREM 4.22. (INVARIANCE OF THE LOCAL DEGREE.) *Let* $C_1$ *be a continuous mapping of a chain* $x^n$ *into* $R_1^n$ *without any roots (relative to the point* $o_1 \in R_1^n$*) on* $\dot{\Delta}x^n$, *so that the degree* $\gamma$ *of* $C_1$ *in the point* $o_1$ *is defined; furthermore, let* $|t_1^n|$ *be an* $n$*-simplex oriented coherently with* $R_1^n$ *and containing both* $o_1$ *and* $C_1\bar{x}^n$ *in its interior. Finally, let* $C_2^1$ *be a topological mapping of* $|t_1^n|$ *into* $R_2^n$.

*Then the local degree of the mapping* $C_2^1C_1$ *of the chain* $x^n$ *in the point* $o_2 = C_2^1o_1$ *is equal to* $\tau\gamma$, *where* $\tau = \pm 1$ *is the degree of* $C_2^1$.

## §5. Vector fields and continuous mappings

**§5.1. The connection between vector fields and mappings.** A system of real continuous functions $v_1, \cdots, v_n$ defined on a subset $M$ of $R^n$ may be interpreted geometrically, first, as a *mapping* $v$ which assigns to every $p \in M$ the point $v(p)$ of $R_1^n$ (which may, of course, coincide with $R^n$) with coordinates $v_1(p), \cdots, v_n(p)$; secondly, it may be interpreted as a *vector field*, consisting of the vectors $v(p)$ (of the space $R_1^n$) assigned to the points $p \in M$ and having the components $v_1(p), \cdots, v_n(p)$.

Hence, the investigation of vector fields in $R^n$ is equivalent to the investigation of mappings. In particular, many of the concepts and theorems of the preceding sections can be interpreted as propositions concerning vector fields. We shall examine this interpretation in only a few special cases.

In the sequel, the mapping $v$ induced by a vector field $\mathfrak{V}(p)$ consisting of vectors $v(p)$ will mean, as we pointed out above, the mapping which assigns to each $p \in M$ the point $v(p)$ with coordinates $v_i(p)$, i.e., the endpoint of the vector $v(p)$ with initial point at the origin.

Let $\mathfrak{z}^{n-1}$ be a proper $(n - 1)$-cycle of a compactum $\Phi \subset R^n$, and suppose that a continuous nonvanishing vector field $\mathfrak{V}$ is prescribed on $\Phi$ [i.e., a continuous vector field $\mathfrak{V}$ such that $v(p) \neq 0$ for all $p \in \Phi$]. We define the *characteristic* of the field $\mathfrak{V}$ on $\mathfrak{z}^{n-1}$ as the index of the origin in $R_1^n$ relative to the cycle $v(\mathfrak{z}^{n-1})$, where $v$ is the mapping of $\Phi$ into $R_1^n$ induced by $\mathfrak{V}$.

In view of Theorem 1.5, this definition may be rephrased as follows:

Assign to every $p \in \Phi$ the ray issuing from the origin of $R_1^n$ which carries the vector $v(p)$; the ray intersects the unit sphere $S^{n-1}$ of $R_1^n$ in a point $C_1(p)$. This defines a continuous mapping $C_1$ of $\Phi$ into $S^{n-1}$. If $S^{n-1}$ is oriented coherently with the prescribed orientation of $R_1^n$, the degree of the mapping $C_1$ of the cycle $\mathfrak{z}^{n-1}$ into the oriented sphere $S^{n-1}$ is the characteristic of the vector field $\mathfrak{V}$ on the cycle $\mathfrak{z}^{n-1}$.

It follows immediately from this definition that the characteristic of a vector field depends only on the direction, but not on the length, of the vectors of the field. Furthermore, the following propositions are easy consequences of the results of the preceding sections:

5.11. Let $\mathfrak{V}_0$ and $\mathfrak{V}_1$ be two nonvanishing vector fields on $\Phi$ with the following property:

There exists a family of vector fields $\mathfrak{V}_\theta$, continuous in $\theta$, $0 \leq \theta \leq 1$, continuous and nonvanishing for each $\theta$ everywhere on $\Phi$.

Then the characteristics of $\mathfrak{V}_0$ and $\mathfrak{V}_1$ are equal.

REMARK. The condition of Theorem 5.11 is satisfied, in particular, if for no $p \in \Phi$ do the vectors $\mathfrak{V}_0(p)$ and $\mathfrak{V}_1(p)$ have opposite directions (Poincaré-Bohl theorem).

5.12. Let $\Gamma$ be a polyhedral domain in $R^n$ (see 2.1, Remark). Every vector field continuous and nonvanishing on $\bar{\Gamma}$ has characteristic 0 on the boundary of $\Gamma$ (Theorem 2.11).

5.13. A vector field continuous and nonvanishing at all points of a solid sphere has characteristic 0 on its bounding sphere (Theorem 2.10).

5.14. The characteristic of the field of exterior normals on an $(n - 1)$-

sphere is equal to $+1$, and the characteristic of the field of interior normals is equal to $(-1)^n$.

*Proof.* Suppose that the sphere $S^{n-1}$ is given by the equation $\sum x_i^2 = 1$ and that the normal vectors have length 1; then the induced mapping $v$ is defined in the case of the exterior normals by the formulas $v_i = x_i$, and in the case of the interior normals by $v_i = -x_i$. The mapping $v$ (considered on the entire space $R^n$) is affine, and its determinant is $+1$ in the first case and $(-1)^n$ in the second case; it therefore has degree $+1$ [or $(-1)^n$] everywhere. Since the degree of $v$ at the origin $o$ (that is, the index of $o$ relative to the image of $S^{n-1}$ under $v$) is also the characteristic of a vector field on this sphere, the assertion is proved.

**§5.2. The index of an isolated zero** $p$ **of a vector field** $\mathfrak{V}$ defined on a subset $M \subset R^n$ which contains $p$ in its interior is defined as follows:

Let $v$ be the mapping of $M \subset R^n$ into $R_1{}^n$ induced by $\mathfrak{V}$. Suppose that $p \in T^n \subset M$ and let $\mathfrak{z}^{n-1}$ be a proper $(n-1)$-cycle with carrier $\Phi \subset T^n$ satisfying

$$\mathfrak{v}(\mathfrak{z}^{n-1}, p) = 1. \tag{5.21}$$

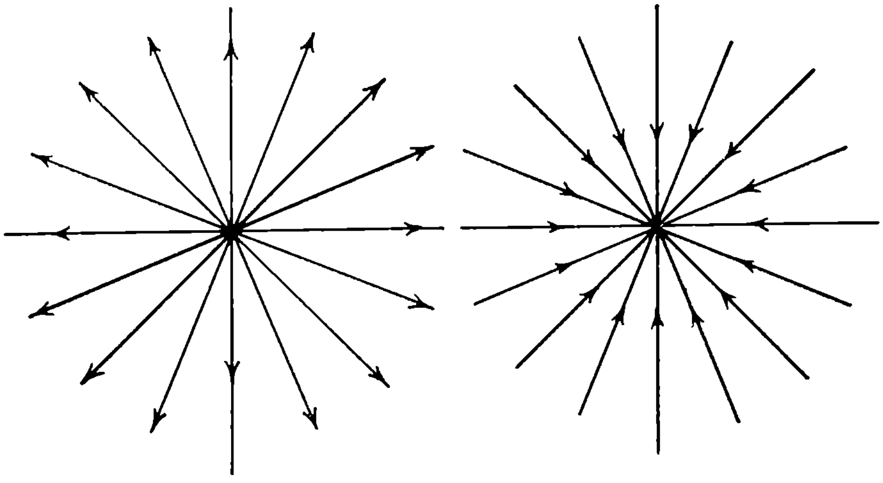
Then, by definition, $\operatorname{ind}_p \mathfrak{V}$ is the characteristic of $\mathfrak{V}$ on $\mathfrak{z}^{n-1}$ (that is, the index of the origin of $R_1{}^n$ relative to the cycle $v\mathfrak{z}^{n-1}$).

In view of Theorem 1.42, this index is independent of the cycle $\mathfrak{z}^{n-1}$ provided it satisfies (5.21). It is simplest of all to put

$$\mathfrak{z}^{n-1} = \Delta t^n,$$

where $p \in |t^n| = T^n \subset M$ and $t^n$ is the orientation of $T^n$ coherent with the orientation prescribed on $R^n$.

Suppose that the cycle $\mathfrak{z}^{n-1}$ with carrier $\Phi \subset T^n$, $\mathfrak{v}(\mathfrak{z}^{n-1}, p) = 1$, to



a b

FIG. 162

have been chosen in this way or in any other. Let $S_1^{n-1}$ be an $(n-1)$-sphere of $R_1^n$ with center $o_1$ and radius 1, oriented coherently with the prescribed orientation of $R_1^n$; we shall call this sphere a sphere of directions. We assign to every $q \in \Phi$ the point $C_1(q) \in S_1^{n-1}$ in which the sphere $S_1^{n-1}$ intersects the ray $o_1\varphi(q)$ parallel to the vector $v(q)$.

Then, by Theorem 1.5, $\mathrm{ind}_p \mathfrak{V}$ *is equal to the degree of the mapping* $C_1$ *of the cycle* $z^{n-1}$ *into the sphere* $S_1^{n-1}$.

REMARK. In the definition of the index of a zero of a vector field, the lengths of vectors play no role; only their directions are important. Therefore, the concept of index can be applied not only to vector fields, but also to *direction fields*. By a direction we mean a ray $(0, \infty)$ of $R_1^n$ or, more precisely the collection of all vectors of the form $av$, with $v \neq 0$ a fixed vector and $a$ an arbitrary positive number. Then instead of the zeros of a vector field we speak of the *singularities* of a direction field; these are the points at which the field is not defined.

Examples of direction fields with isolated singularities are given in Figs. 162–165. In each figure the singularity is the origin. In Figs. 162 and 163 the index is $+1$, in Figs. 164 and 165 it is, respectively, $+3$ and $-1$. All the figures are plane figures.

### §5.3. Vector fields on a solid sphere.

THEOREM 5.3. *Let* $\bar{E}^n$ *be a closed solid n-sphere and let* $S^{n-1} = z^{n-1}$ *be its bounding* $(n-1)$*-sphere. If* $\mathfrak{V}$ *is a vector field continuous and nonvanishing on* $\bar{E}^n$, *there exists at least one point of* $S^{n-1}$ *at which* $\mathfrak{V}$ *assumes the direction of the exterior normal to* $S^{n-1}$, *and at least one point of* $S^{n-1}$ *at which* $\mathfrak{V}$ *assumes the direction of the interior normal to* $S^{n-1}$.

*Proof.* According to 5.13, the characteristic of $\mathfrak{V}$ on $z^{n-1}$ is 0. If $\mathfrak{W}$ is a field on $z^{n-1}$ with non-zero characteristic, there exists, in view of the Remark to 5.11, a point $p \in S^{n-1}$ at which $v(p)$ and $w(p)$ have opposite directions. By 5.14, we may choose for $\mathfrak{W}$ the field of exterior or interior

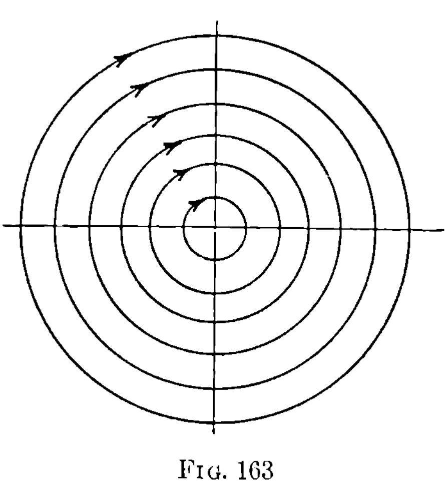


FIG. 163

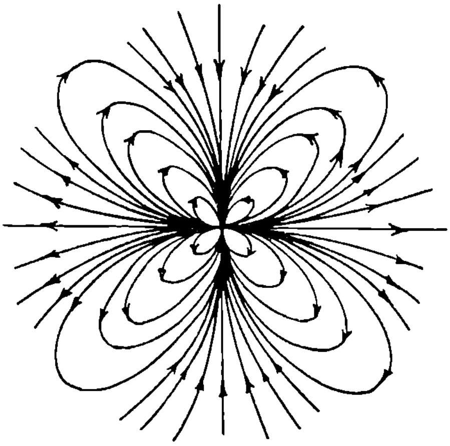


FIG. 164

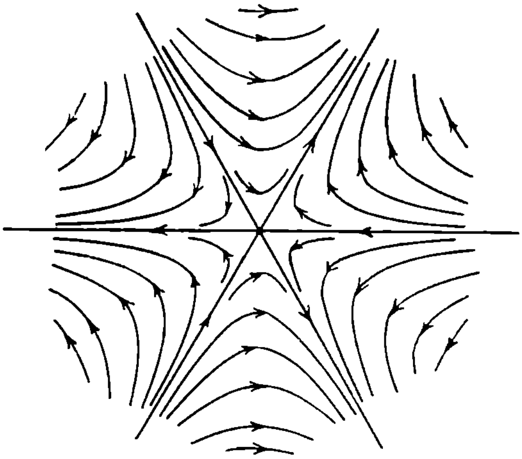


FIG. 165

normals; consequently, the field $\mathfrak{B}$ on $S^{n-1}$ contains at least one interior normal and at least one exterior normal.

**§5.4. Brouwer's fixed point theorem for an $n$-cell.** Let $C$ be a continuous mapping of a solid $n$-sphere $\bar{E}^n \subset R^n$ into $R^n$. To each point $p \in \bar{E}^n$ assign the vector $v(p) = pC(p)$; this continuous vector field vanishes at precisely those points of $\bar{E}^n$ at which $Cp = p$, that is, at the fixed points of $C$. If C has no fixed points, the conditions of Theorem 5.3 are satisfied and, consequently, there exists at least one point $p$ on the bounding $(n - 1)$-sphere $S^{n-1}$ of $\bar{E}^n$ at which the vector $v(p)$ has the direction of the exterior normal. Hence, the point $Cp$ is contained in the exterior of $\bar{E}^n$. Therefore,

THEOREM 5.40. *Let $\bar{E}^n$ be a solid n-sphere in $R^n$ and let $S^{n-1}$ be its bounding $(n - 1)$-sphere. Every continuous mapping $C$ of $\bar{E}^n$ into $R^n$ such that $CS^{n-1} \subseteq \bar{E}^n$ has at least one fixed point in $\bar{E}^n$.*

In particular, every continuous mapping of $\bar{E}^n$ into itself has a fixed point. It is clear that every set homeomorphic to $\bar{E}^n$, that is, every closed $n$-cell, also has this property. We have, therefore,

5.4. BROUWER'S FIXED POINT THEOREM FOR A CLOSED $n$-CELL. *Every continuous mapping of a closed n-cell into itself has at least one fixed point.*

### §5.5. Vector fields on spheres of even dimension.

FIXED POINT THEOREM. Let $n$ be odd and let $S^{n-1}$ be an $(n-1)$-sphere in $R^n$. According to Theorem 5.14 the field $\mathfrak{N}_e$ of exterior and the field $\mathfrak{N}_i$ of interior normals to $S^{n-1}$ have different characteristics on $S^{n-1}$. It follows that if $\mathfrak{V}$ is a field on $S^{n-1}$, then its characteristic is different from the characteristic of at least one of the fields $\mathfrak{N}_e$ and $\mathfrak{N}_i$. Hence, by the Remark following Theorem 5.11 there must exist a point on $S^{n-1}$ at which the vector $v$ has a direction opposite to one of the normal vectors, that is, the vector $v$ is itself a normal vector. We have thus proved

THEOREM 5.51. *Every continuous nonvanishing vector field on $S^{n-1}$, where $n-1$ is even, contains at least one vector normal to $S^{n-1}$ in $R^n$.*

A special case is

THEOREM 5.510. (THE POINCARÉ-BROUWER THEOREM.) *There does not exist a continuous field of tangent directions on a sphere $S^n$ of even dimension.*

REMARK. Hence, if a porcupine curls up into a ball, its needles cannot lie smoothly—some of them will immediately stick out. For this reason, Hilbert called Theorem 5.510 "the porcupine theorem".

If $n$ is odd, however, the formulas

$$v_{2j-1} = -x_{2j}, \qquad v_{2j} = x_{2j-1} \qquad [j = 1, 2, \cdots, \tfrac{1}{2}(n+1)],$$

for instance, define a continuous field of tangent directions with components $v_i$ on the sphere $S^n$, given by the equations $\sum_{i=1}^{n+1} x_i^2 = 1$.

From Theorem 5.51 it is easy to derive

THEOREM 5.52. *For every continuous mapping of an n-sphere $S^n$, $n$ even, into itself there exists at least one fixed point or at least one point which is mapped into its diametrically opposite point (or both).*

*Proof.* If a mapping $C$ of $S^n$ into itself has no fixed points, the vectors $pC(p)$ form a continuous nonvanishing vector field on $S^n$. Hence, by Theorem 5.51 there exists a point $p$ at which the vector $pC(p)$ coincides in direction with a normal to $S^n$. But this is possible only if the point $Cp$ is diametrically opposite to $p$.

If $n$ is odd, however, the formulas

$$\bar{x}_{2j-1} = -x_{2j}, \qquad \bar{x}_{2j} = x_{2j-1} \qquad [j = 1, 2, \cdots, \tfrac{1}{2}(n+1)]$$

define a mapping of the sphere $S^n$, given by $\sum x_i^2 = 1$, into itself which has neither a fixed point nor maps any point into its diametrically opposite point.

**§5.6. Reflections of spheres; another fixed point theorem.** A reflection of $R^n$ about a plane $R^r \subseteq R^n$, $0 \leq r \leq n$, is a mapping of $R^n$ onto itself defined as follows: choose a system of rectangular coordinates $x_i$ ($1 \leq i \leq n$) in $R^n$ so that $R^r$ is given by the equations

$$x_1 = x_2 = \cdots = x_{n-r} = 0.$$

Then a reflection $C$ about $R^r$ is defined by the formulas

$$(5.61) \qquad \begin{aligned} x_i' &= -x_i & (1 \leq i \leq n - r), \\ x_i' &= x_i & (n - r + 1 \leq i \leq n). \end{aligned}$$

This mapping maps every $(n - 1)$-sphere with center in $R^r$ into itself.

THEOREM 5.6. *A reflection $C$ of $R^n$ about a plane $R^r$ restricted to a sphere $S^{n-1}$ with center $o = (0, \cdots, 0)$ is a mapping of $S^{n-1}$ (onto itself). This mapping is homotopic to the identity mapping in $R^n \setminus o$ if, and only if,* $r \equiv n \pmod 2$.

*Proof.* The determinant of the mapping $C$, defined by the formulas (5.61), is equal to $(-1)^{n-r}$. Therefore (see the proof of 5.14),

$$\mathfrak{v}(Cz^{n-1}, o) = (-1)^{n-r},$$

where $z^{n-1}$ is the sphere $S^{n-1}$ oriented coherently with the prescribed orientation of $R^n$. On the other hand, $\mathfrak{v}(z^{n-1}, o) = +1$. Hence, if $r \not\equiv n \pmod 2$, so that $(-1)^{n-r} = -1$, then the cycles $z^{n-1}$ and $Cz^{n-1}$ cannot be homotopic. Consequently, $C$ is not homotopic to the identity in $R^n \setminus o$.

If $r \equiv n \pmod 2$, then the formulas

$$\left.\begin{aligned} x_{2i-1}' &= x_{2i-1}\cos\theta\pi - x_{2i}\sin\theta\pi, \\ x_{2i}' &= x_{2i-1}\sin\theta\pi - x_{2i}\cos\theta\pi \end{aligned}\right\} \quad [1 \leq i \leq \tfrac{1}{2}(n - r)],$$

$$x_i' = x_i \qquad (n - r + 1 \leq i \leq n)$$

define a family of mappings $C_\theta$, $0 \leq \theta \leq 1$, which is the required deformation of the identity mapping into $C$ (this deformation is even a displacement of $S^{n-1}$ into itself).

The following special case deserves particular attention:

THEOREM 5.60. *A reflection of a sphere $S^{n-1}$ about its center $o$ is homotopic to the identity in $R^n \setminus o$ if, and only if, $n$ is even.*

Theorem 5.6 further implies

THEOREM 5.61. *If $n$ is odd, then every mapping $C'$ of $S^{n-1}$ into itself which is homotopic to the identity in $R^n \setminus o$ has a fixed point.*

Indeed, if $C'$ has no fixed points, then $C'$ and the reflection $C$ of $S^{n-1}$ about its center $o$ satisfy the hypotheses of the Poincaré-Bohl theorem; hence $C'$ is homotopic to $C$ in $R^n \setminus o$, and by 5.60 is not homotopic to the identity.

## §5.7. Exercises.

1°. Let

$$y_i = f_i(x_1, x_2, \cdots, x_n) \qquad (1 \leq i \leq n)$$

be a system of $n$ real-valued continuous functions of $n$ real variables with continuous partial derivatives in a neighborhood of a point $o = (x_1^0, x_2^0, \cdots, x_n^0) \in R^n$. Suppose that the Jacobian of this system of functions is different from zero at the point $o$. Then it is known from analysis that this system of functions defines a topological mapping $C$ of a neighborhood of the point $o$ onto a neighborhood of the point

$$o' = (y_1^0, y_2^0, \cdots, y_n^0), \qquad \text{where } y_i^0 = f_i(x_1^0, \cdots, x_n^0).$$

Prove that *the mapping $C$ has degree 1 or $-1$ in the point $o'$, depending on the sign of the Jacobian.*

REMARK. The continuity of the partial derivatives is not necessary; only the weaker requirement that the total differential exist is necessary: the mapping $C$ can then be replaced by an affine mapping which approximates it near the point $o$ and the problem can then be solved. It is well to note that the solution of the problem does not require that $C$ be a topological mapping.

2°. The function

$$y = ax^n$$

of a complex variable $x$ defines a mapping of the complex sphere onto itself; the degree of the mapping is $n$. If the sphere is thought of as the plane compactified by the addition of a point at infinity, then the local degree of the mapping in the point $o$ is $n$. Using this result, prove the more general proposition:

Suppose that the power series

$$y = a_n x^n + a_{n+1} x^{n+1} + \cdots \qquad (a_n \neq 0)$$

converges in a neighborhood of $x = 0$. *Then it represents a mapping of this neighborhood onto some other neighborhood of $x = 0$, whose degree in the point $x = 0$ is $n$.*

3°. ANALYTIC FORMULATION OF THEOREMS 5.3 AND 5.51.

THEOREM 5.3. *Let $v_i(x_1, \cdots, x_n)$ $(1 \leq i \leq n)$ be a system of continuous functions defined on the solid sphere $\sum_{i=1}^n x_i^2 \leq 1$, which are not all zero at the same point of the sphere. Then there exist a positive number $\lambda$ and a negative number $\nu$ with the property that each of the systems of equations*

$$v_i(x_1, \cdots, x_n) = \lambda x_i \qquad (1 \leq i \leq n)$$

*and*

$$v_i(x_1, \cdots, x_n) = \nu x_i \qquad (1 \le i \le n)$$

*has a solution on the sphere* $\sum_{i=1}^{n} x_i^2 = 1$.

THEOREM 5.51. *Suppose that a set of continuous functions* $v_i(x_1, \cdots, x_n)$ $(1 \le i \le n)$ *is defined on the sphere* $\sum_{i=1}^{n} x_i^2 = 1$, $n$ *odd. Then there exists a real number* $\lambda$ *such that the system of equations*

$$v_i(x_1, \cdots, x_n) = \lambda x_i \qquad (1 \le i \le n)$$

*has a solution on the sphere* $\sum_{i=1}^{n} x_i^2 = 1$.

4°. APPLICATION OF THEOREM 5.4 TO THE PROOF OF FROBENIUS' THEOREM ON THE EXISTENCE OF EIGENVALUES OF MATRICES. An eigenvalue of a square matrix $(a_{ij})$ is a root of its characteristic equation

$$\det (a_{ij} - \lambda \delta_{ij}) = 0.$$

THEOREM OF FROBENIUS. *Every square matrix* $(a_{ij})$, *whose elements are nonnegative real numbers, has at least one nonnegative real eigenvalue.*

*Proof.* Consider the mapping of the projective $(n - 1)$-space $P^{n-1}$ defined by the equations

$$x_i' = \sum_j a_{ij} x_j \qquad (1 \le i \le n).$$

The points of $P^{n-1}$ all of whose coordinates $x_k$ are nonnegative form a closed $(n - 1)$-simplex in $P^{n-1}$. Since $a_{ij} \ge 0$, the mapping defined above maps this simplex into itself and therefore has at least one fixed point. The coordinates of a fixed point satisfy the equations

$$\lambda x_i = \sum_{j=1}^{n} a_{ij} x_j$$

for some real $\lambda \neq 0$, which is therefore an eigenvalue of the matrix. Since both the $a_{ij}$ and $x_j$ are nonnegative, $\lambda$ is also nonnegative, which was to be proved.

These examples illustrate how topological fixed point theorems can be used to prove existence theorems. It is also easy to see that the proofs of existence theorems for various functional (in particular, integral) equations require correspondingly stronger fixed point theorems (for example, the fixed point theorem for continuous mappings of convex compacta in Hilbert space).

## §6. Classification of continuous mappings of an $n$-sphere into an $n$-sphere

**§6.1. Hopf's theorem; introductory remarks.** We already know that two homotopic continuous mappings of one orientable (and oriented) closed $n$-dimensional pseudomanifold into another have the same degree (XI, Theorem 6.41). This is, of course, also true in the special case of an $n$-sphere.

In this special case, however, the converse is also true: two continuous mappings of an $n$-sphere into an $n$-sphere, which have the same degree, are homotopic. Thus, we arrive at the following remarkable result, which we shall prove in this section:

HOPF'S THEOREM 6.1. *Let* $S_0^n$ *and* $S^n$ $(n \geq 1)$ *be two n-spheres with prescribed orientations. Two continuous mappings of* $S_0^n$ *into* $S^n$ *are homotopic if, and only if, they have the same degree.*

REMARK 1. This theorem was proved by Brouwer in 1912 for $n = 2$; the general result was obtained by Hopf in 1926.

The proof given here is due to Hopf and is taken in part from Alexandroff-Hopf, *Topologie I* (Chapter 13, §1) and in part from Hopf's lectures, given at the University of Berlin in 1926–1927. It displays profound geometric insight.

Another very simple proof of Hopf's theorem was given by Whitney (see Whitney [a]).

REMARK 2. It follows from VIII, 5.24 that for arbitrary $n \geq 1$ and arbitrary integer $\gamma$ there exist mappings of one $n$-sphere into another having degree $\gamma$.

The proof of Hopf's theorem is preceded by several auxiliary propositions.

DEFINITION 6.11. If a mapping of a set $X$ into a set $Y$ maps $X$ onto a single point of $Y$, the mapping will be called a *constant mapping.* If a mapping $f$ of $X$ into $Y$ is homotopic to a constant mapping, we shall say that $f$ is homotopic to zero and write: $f \simeq 0$. If $C$ is homotopic to $C_1$, we write: $C \simeq C_1$.

REMARK 3. Two constant mappings of an $n$-sphere (or in general of a compactum) into an $n$-sphere $S^n$, $n \geq 1$, are homotopic. This follows from the fact that for $n \geq 1$ a pair of points of $S^n$ can be connected by a simple arc.

6.12. *Let* $\bar{E}_1^n$ *be a polyhedron homeomorphic to a closed n-simplex. A mapping* $C_1$ *of the boundary* $\dot{E}_1^n$ *of* $\bar{E}_1^n$ *into a compactum* $\Phi$ *is homotopic to zero if, and only if,* $C_1$ *can be extended to all of* $\bar{E}_1^n$.

It is clearly sufficient to prove this proposition on the supposition that $\bar{E}_1^n$ is a closed $n$-simplex $\bar{T}_1^n$.

Let $C_\theta$, $0 \leq \theta \leq 1$, be a deformation of $C_1$ into the constant mapping $C_0$ which assigns to all the points of $S_1^{n-1} = \bar{T}_1^n \setminus T_1^n$ a single point $c_0 \in \Phi$. Let $p_0$ be a point in the interior of $T_1^n$ and set $C(p_0) = c_0$. Let $p$ be an arbitrary point of $T_1^n$ distinct from $p_0$. Denote by $p'$ the point of intersection of the ray $p_0p$ with $S_1^{n-1}$ and by $\theta$ the ratio in which $p$ divides the segment $p_0p'$. Then set $C(p) = C_\theta(p')$.

The mapping $C$ of the closed simplex $\bar{T}_1^n$ obtained in this way is the required one. Conversely, if $C$ is an extension of $C_1$ to all of $\bar{T}_1^n$, then the mapping $C_1$ of the boundary $S_1^{n-1}$ of $T_1^n$ is homotopic to zero.

To see this, let $p_0$ be the centroid of $T_1^n$. Denote by $\bar{T}_\theta^n$ the image of $\bar{T}_1^n$ under the similitude with center $p_0$ and coefficient $\theta$, $0 \leq \theta \leq 1$. For each $p \in S_1^{n-1}$ denote by $p_\theta$ the point of intersection of the ray $p_0p$ with the boundary of the simplex $T_\theta^n$ and set $C_\theta(p) = C(p_\theta)$. The family of mappings $C_\theta$, $0 \leq \theta \leq 1$, defines a deformation of the mapping $C_1$ of the sphere $S_1^{n-1}$ into the constant mapping $C_0'$ which assigns to every point $p \in S_1^{n-1}$ the single point $C(p_0) \subset \Phi$.

6.13. *Let* $C_0$ *and* $C_1$ *be two continuous mappings of a compactum* $\Phi$ *into* $S^n$. *If* $S^n$ *contains a nonempty open set* $\Gamma$ *such that* $C_0^{-1}(q) = C_1^{-1}(q)$ *for every* $q \in \Gamma$, *then* $C_0 \simeq C_1$.

*Proof.* Choose a definite point $u \in \Gamma$ and think of $S^n$ as $R^n$ compactified by the addition of the point $u$ as the point at infinity. Suppose that one of $C_0(p)$, $C_1(p)$ is different from $u$ for some $p \in \Phi$. Then $C_0(p)$, $C_1(p)$ are both different from $u$ (since $u \in \Gamma$). For such a point $p$ define $C_\theta(p)$, $0 \leq \theta \leq 1$, as the point which divides the segment $C_0(p)C_1(p)$ in the ratio $\theta:(1 - \theta)$. If $C_0(p) = u$, set $C_\theta(p) = C_0(p) = u$ for all $\theta$. It is clear that $C_\theta$ is continuous for every $\theta$ if $p \notin C_0^{-1}(\Gamma) = C_1^{-1}(\Gamma)$. If $p \in C_0^{-1}(\Gamma)$, the set $C_0^{-1}(\Gamma)$ is a neighborhood of $p$ at all of whose points $C_0$ and $C_1$, and hence also the mappings $C_0$ and $C_\theta$ for all $\theta$, coincide, so that $C_\theta$ is also continuous at all points $p \in C_0^{-1}(\Gamma)$. Finally, the family of mappings $C_\theta$ is continuous in $\theta$ and defines a deformation of $C_0$ into $C_1$.

6.14. *Suppose that* $n \geq 2$; *let* $C$ *be a continuous mapping of a sphere* $S_0^n$ *into a sphere* $S^n$ *and suppose that* $p \in S^n$. *Then there exists a continuous mapping* $C_\nu$ *of* $S_0^n$ *into* $S^n$ *homotopic to* $C$ *and with the following property*: $C_\nu^{-1}(p)$ *is contained in an* $n$*-simplex* $T_0^n$ (*of a triangulation of* $S_0^n$) *whose image* $C_\nu(T_0^n)$ *is a proper subset of* $S^n$.

*Proof.* Since every continuous mapping of $S_0^n$ into $S^n$ is homotopic to any of its simplicial approximations, we may suppose that $C$ is a simplicial mapping of a triangulation $K_0^n$ of $S_0^n$ into a triangulation $K^n$ of $S^n$. We are further justified in assuming that $p$ is contained in an $n$-simplex of $K^n$.

Finally, let $u$ be a point different from $p$, which is also contained in an $n$-simplex of $K^n$. The inverse image of each of the points $p$ and $u$ consists of a *finite* number of points of $\| K_0^n \| = S_0^n$. (*It is for this reason alone that we required the mapping* $C$ *to be simplicial.*)

We again think of $S^n$ as $R^n$ compactified by the addition of $u$ as the point at infinity, and $S_0^n$ as $R_0^n$ compactified by the addition of a point $u_0 \notin C^{-1}(p) \cup C^{-1}(u)$.

Let $\bar{T}_0^n$ be a closed $n$-simplex of $R_0^n$ such that $\bar{T}_0^n \cap C^{-1}(u) = 0$ (we may always assume that $T_0^n$ is a simplex of a triangulation of $S_0^n$). It then follows that

$$C(\bar{T}_0^n) \subset R^n.$$

Suppose that $p_0 \in C^{-1}(p)$ is in the complement of $T_0^n$. Since $n \geq 2$, $T_0^n$ contains a point $p_0'$ such that the segment $p_0'p_0 \subset S_0^n \setminus C^{-1}(u)$. There is therefore an $n$-simplex $T_0'^n \subset S_0^n$ containing the segment $p_0'p_0$, at a positive distance from the finite set $C^{-1}(u)$ and having no point of $C^{-1}(p)$ on its boundary.

We have

$$C(\bar{T}_0'^n) \subset R^n = S^n \setminus u.$$

We now define a mapping $C_1$ by setting $C_1 = C$ on $S_0^n \setminus T_0'^n$, by setting $C_1(p_0') = p$ and finally, if $q_0$ is an arbitrary point of the boundary of $T_0'^n$, by defining $C_1$ on the segment $p_0'q_0$ as the linear mapping of this segment onto the segment $pC_1(q_0)$ of $R^n$. Since $C_1$ differs from $C$ only in $T_0'^n$ and since $C(\bar{T}_0'^n)$ and $C_1(\bar{T}_0'^n)$ are closed and bounded subsets of $R^n$, the set

$$\Gamma = S^n \setminus [C(\bar{T}_0'^n) \cup C_1(\bar{T}_0'^n)]$$

containing $u$ satisfies the hypotheses of 6.13; hence $C \simeq C_1$.

It is clear that $C_1^{-1}(p) \cup C_1^{-1}(u)$ is a finite set. The only point of $C_1^{-1}(p)$ contained in $\bar{T}_0'^n$ is the point $p_0'$, which is contained in $T_0^n$. The set $C_1^{-1}(p)$ is finite. Therefore, if $C$ is replaced by $C_1$ the inverse images $C_1^{-1}(u)$ and $C_1^{-1}(p)$ of $u$ and $p$ remain finite and the number of points of $C_1^{-1}(p)$ in the complement of $T_0^n$ is less than the number of points of $C^{-1}(p)$ in the complement of $T_0^n$. It is true that $C_1$ may not be simplicial; however, the only relevant property of simplicial mappings is for us the finiteness of the inverse images of $p$ and $u$, as we mentioned above, and this property $C_1$ does have. Finally, the set

$$C_1(\bar{T}_0^n) \subset R^n$$

is a proper subset of $S^n$.

Repeating the same process a finite number of times, say $\nu$ times, if necessary, we arrive at a mapping $C_\nu$ homotopic to $C$, such that $C_\nu^{-1}(p)$ is contained in $T_0^n$ and $C_\nu(\bar{T}_0^n) \subset R^n \neq S^n$. This completes the proof.

**§6.2. Mappings of degree zero.** We shall prove the following theorem:

6.21$_n$. *A continuous mapping of an n-sphere $S_0^n$ into an n-sphere $S^n$ of degree zero is homotopic to zero.*

Remark. In proving Theorem 6.21$_n$ we shall assume that $S_0^n$ is *the boundary of an $(n + 1)$-simplex $T_0^{n+1}$*, which is, of course, topologically equivalent to an $n$-sphere.

*Proof of Theorem* $6.21_n$. The proof is by induction and also involves the following theorems:

$6.22_n$. *Let* $t_0^{n+1}$ *be the orientation of* $T_0^{n+1}$ *coherent with that of the prescribed orientation of* $R^{n+1}$ *(the boundary of* $T_0^{n+1}$ *is* $S_0^n$*). Suppose that* $o \in R^{n+1}$ *and that* $C$ *is a continuous mapping of* $S_0^n$ *into* $R^{n+1} \setminus o$. *If* $\mathfrak{v}(C\Delta t_0^{n+1}, o) = 0$, *then* $C$ *can be extended to a mapping of the closed simplex* $\bar{T}_0^{n+1}$ *into* $R^{n+1} \setminus o$.

$6.23_n$. *Retaining the notation of Theorem* $6.22_n$, *let us suppose that* $C$ *is a mapping of* $S_0^n$ *into a sphere* $S^n \subset R^{n+1}$ *with center* $o$ *and that the degree of* $C$ *is zero. Then* $C$ *can be extended to a mapping of the closed simplex* $\bar{T}_0^{n+1}$ *into* $S^n$.

These theorems will all be proved together by proving the following four propositions:

1°. Theorems $6.21_0$ and $6.21_1$ are true.

2°. $6.21_n$ implies $6.23_n$, $n \geq 0$.

3°. $6.23_n$ implies $6.22_n$, $n \geq 0$.

4°. $6.22_n$ implies $6.21_{n+1}$, $n \geq 1$.

*Proof of* 1°. Theorem $6.21_0$ is true since a mapping of a 0-sphere into a 0-sphere of degree zero is a constant mapping (VIII, 5.2, Remark 4). Theorem $6.21_1$ is a special case of II, Theorem 2.5 and XI, 6.4, Remark on the case $n = 1$.

*Proposition* 2° follows from 6.12.

*Proof of* 3°. On the supposition that every mapping $C$ of degree zero of $S_0^n$ into a sphere $S^n$ with center $o$ can be extended to a mapping of $\bar{T}_0^{n+1}$ into $S^n$, we are to prove that every mapping $C$ of $S_0^n$ into $R^{n+1} \setminus o$ for which $\mathfrak{v}(C\Delta t_0^{n+1}, o) = 0$ can be extended to a mapping of $\bar{T}_0^{n+1}$ into $R^{n+1} \setminus o$. We divide this extension problem into two problems: we introduce into $R^{n+1}$ a system of "polar coordinates" consisting of a pole $o$ and the unit sphere $S^n$ with center $o$; the polar coordinates of a point $p \in R^{n+1} \setminus o$ are the radius vector $r(p)$ of $p$ (i.e., the distance of $p$ from $o$) and the projection of $p$ on $S^n$ (i.e., the point of intersection of the ray $op$ with $S^n$), which we denote by $\varphi(p)$. Then the extension problem reduces to:

a) The extension of the positive function $rC(p)$ defined on $S_0^n$ to a function defined on all of $\bar{T}_0^{n+1}$.

b) The extension of the mapping $\varphi C$ of $S_0^n$ into $S^n$ to a mapping of $\bar{T}_0^{n+1}$ into $S^n$.

Problem a) is easily solved by applying the Brouwer-Urysohn theorem on the extension of continuous functions (I, Theorem 4.22); in extending a positive function we may always require that the extended function also be positive. The latter remark, which is true in general (see, e.g., Alexandroff-Hopf, Ch. 1, §6, Zusatz on page 78), follows in this case from the fact that $rC(p)$ is continuous and positive on a compact set.

The solution of Problem b) is already contained in our hypothesis (Theorem $6.23_n$), since the degree of the mapping $\varphi C$ of $S_0^n$ into $S^n$ is equal to the index of $o$ relative to $C\Delta t_0^{n+1}$ (Theorem 1.51), that is, zero.

*Proof of* 4°. Given a mapping $C$ of degree zero of $S_0^{n+1}$ into $S^{n+1}$, $n \geq 1$, we are required to find a deformation of $C$ into a constant mapping. In order to do this, it is sufficient to deform $C$ into a mapping $C_1$ which maps $S_0^{n+1}$ onto a proper subset of $S^{n+1}$. Since every proper subset of $S^{n+1}$ is contractible to a point, $C_1$ can be deformed into a constant mapping.

Since $n \geq 1$, we are justified in applying Lemma 6.14 and in assuming that $C$ satisfies the following condition:

There is a point $o \in S^{n+1}$ such that $C^{-1}(o)$ is contained in an $(n + 1)$-simplex $T_0^{n+1}$ the image of whose closure [i.e., the set $C(\bar{T}_0^{n+1})$] is a proper subset of $S^{n+1}$.

We make this assumption and denote by $u$ a definite point of

$$S^{n+1} \setminus C(\bar{T}_0^{n+1}).$$

Next, we again think of $S^{n+1}$ as $R^{n+1}$ compactified by the addition of $u$ as the point at infinity, so that

$$C(\bar{T}_0^{n+1}) \subset S^{n+1} \setminus u = R^{n+1}.$$

Since $C^{-1}(o) \subset T_0^{n+1}$, we can apply Theorem 3.3 (in which we replace $n$ by $n + 1$). This gives

$$\mathfrak{v}(C\Delta t_0^{n+1}, o) = \gamma_C = 0,$$

where $t_0^{n+1}$ is the orientation of $T_0^{n+1}$ coherent with the orientation of $S_0^{n+1}$.

It follows by Theorem $6.22_n$ (which we suppose to be correct) that there exists a mapping $C_1$ of $\bar{T}_0^{n+1}$ into $R^{n+1} \setminus o \subset S^{n+1} \setminus (o \cup u)$ which coincides with $C$ on $\bar{T}_0^{n+1} \setminus T_0^{n+1}$. Set $C_1(p) = C(p)$ for arbitrary

$$p \in S_0^{n+1} \setminus \bar{T}_0^{n+1}.$$

Now $C_1$ is already a mapping of $S_0^{n+1}$ into $S^{n+1} \setminus o$, i.e., onto a proper subset of $S^{n+1}$. The open set

$$\Gamma = S^{n+1} \setminus [C(\bar{T}_0^{n+1}) \cup C_1(\bar{T}_0^{n+1})]$$

is nonempty (it contains $u$), and $C^{-1}(q) = C_1^{-1}(q)$ for every $q \in \Gamma$; hence, by Lemma 6.13, $C \simeq C_1$. This proves 4° and completes the proof of Theorems 6.21, 6.22 and 6.23.

### §6.3. Reduction of Hopf's theorem to an extension theorem.

EXTENSION THEOREM. *Let $\bar{T}_0^n$ be a closed simplex contained in a simplex $T_1^n \subset R^n$. We denote the boundaries of $T_0^n$ and $T_1^n$ by $S_0^{n-1}$ and $S_1^{n-1}$, respectively. Let us denote the polyhedral domain $T_1^n \setminus \bar{T}_0^n$ by $\Gamma$ and orient* *it coherently with the prescribed orientation of* $R^n$; *the boundary of the oriented domain* $\Gamma$ (*in some triangulation* $K$) *we denote by* $z^{n-1}$. *Let* $C$ *be a continuous mapping of the set*

$$S_0^{n-1} \cup S_1^{n-1} = \bar{z}^{n-1} = \bar{\Gamma} \setminus \Gamma$$

*into an* $(n - 1)$*-sphere* $S^{n-1}$ *of degree zero on the cycle* $z^{n-1}$.

*Then* $C$ *can be extended to all of* $\bar{\Gamma}$.

Let us assume that the extension theorem is true and derive Hopf's theorem from it.

Suppose that $o$ is a definite point of $T_0^n$. We are given mappings $C_0$ and $C_1$ of the same degree of $S_0^{n-1}$ into $S^{n-1}$.

We construct a mapping $C$ of $S_0^{n-1} \cup S_1^{n-1} = \bar{\Gamma} \setminus \Gamma$ into $S^{n-1}$ as follows: By definition, $C = C_0$ on $S_0^{n-1}$.

On $S_1^{n-1}$, $C$ is defined so that if $p' \in S_1^{n-1}$ and $p$ is the point of intersection of the ray $op'$ with $S_0^{n-1}$, then $C(p') = C_1(p)$. (Briefly, the mapping $C$ on $S_1^{n-1}$ is the mapping $C_1$ carried over from $S_0^{n-1}$ onto $S_1^{n-1}$ by means of a central projection from $o$ as center.) Using the extension theorem, we extend $C$ to $\bar{\Gamma}$.

Now let $\theta$ be a real number, $0 \leq \theta \leq 1$. We define a continuous mapping $C_\theta$ of $S_0^{n-1}$ into $S^{n-1}$ as follows: For every $p_0 \in S_0^{n-1}$ denote by $p_1$ the point of intersection of the ray $op_0$ with $S_1^{n-1}$ and by $p_\theta$ the point of the segment $p_0p_1$ which divides it in the ratio $\theta:(1 - \theta)$. Now set $C_\theta(p_0) = C(p_\theta)$. (If, for instance, $T_1^n$ is the result of a similarity mapping of $T_0^n$ with center $o$ and coefficient 2, and $T_\theta^n$ is the result of a similarity mapping with coefficient $1 + \theta$, then $C_\theta$ is obtained by carrying the mapping $C$ of the boundary of $T_\theta^n$ over to $S_0^{n-1}$ by means of a central projection with center $o$.)

The family of continuous mappings $C_\theta$, $0 \leq \theta \leq 1$, of $S_0^{n-1}$ into $S^{n-1}$ is obviously a deformation of $C_0$ into $C_1$. This proves Hopf's theorem. We must now prove the extension theorem.

**§6.4. Proof of the extension theorem.** We first state a few elementary facts.

1°. If $p_0$ is a point of a simplex $T_0^n$ and $C$ is a mapping of

$$S_0^{n-1} = \bar{T}_0^n \setminus T_0^n$$

into a sphere $S^{n-1}$, then $C$ can be extended to all of $\bar{T}_0^n \setminus p_0$.

To see this, let $p$ be an arbitrary point of $\bar{T}_0^n$ distinct from $p_0$. If $p'$ is its projection from $p_0$ on $S^{n-1}$, we set $C(p) = C(p')$.

REMARK. Proposition 1° remains true if the closed simplex $\bar{T}_0^n$ is replaced by an arbitrary homeomorphic image $\bar{E}^n$ of itself.

2°. Let $\bar{T}_0^n$ and $\bar{T}_*^n$ be disjoint closed simplexes contained in the interior of a simplex $T^n$. Suppose that the orientation $t^n$ of $T^n$ is prescribed. If the boundary of the oriented polyhedral domain

$$\Gamma = T^n \setminus (\bar{T}_0^n \cup \bar{T}_*^n)$$

(in some triangulation $K_0^n$) is $z_0^{n-1}$, then every continuous mapping $C$ of $\bar{\Gamma}$ into a sphere $S^{n-1}$ has degree zero on $z_0^{n-1}$.

To prove this, we note that in view of 1.51 the degree of $C$ on $z_0^{n-1}$ is equal to the index of the center of $S^{n-1}$ relative to the cycle $Cz_0^{n-1}$. If this index were different from zero, the center of the sphere would be contained, according to Theorem 2.11, in the image of $\bar{\Gamma}$. This cannot be true, since $C$ is a mapping into $S^{n-1}$.

We can now turn to the proof of the extension theorem. In the sequel we retain the notation of 6.3. Let $K$ be a triangulation of the polyhedron $\bar{\Gamma}$. The mapping $C$ is defined at the vertices of $K$ contained in $S_1^{n-1} \cup S_0^{n-1}$. Define $C$ arbitrarily at all the remaining vertices of $K$.

Suppose $m < n$. We assume that $C$ has been defined on all the elements of $K$ of dimension $\leq m - 1$. We shall now define $C$ on all the $m$-elements of $K \setminus (S_1^{n-1} \cup S_0^{n-1})$. To do so, we first note that $C$ is homotopic to zero on the boundary $\bar{T}^m \setminus T^m$ of an arbitrary $m$-element $T^m$ of $K$. This is true because it is homotopic to a simplicial approximation of $C$, which maps $\bar{T}^m \setminus T^m$ onto a nowhere dense subset of $S^{n-1}$.

Since $C$ is homotopic to zero, it can be extended to all of the simplex $T^m$ (Lemma 6.12). Hence $C$ can be extended, step by step, to all the simplexes of $K$ of dimension $\leq n - 1$.

We now choose a point $p_i$ in each simplex $T_i^n \in K$ and extend the mapping $C$ of the boundary of $T_i^n$ to all the sets $T_i^n \setminus p_i$ in accordance with 1°. The mapping $C$ has now been extended to all of $\bar{\Gamma}$ except for a finite number of exceptional points $p_1, \cdots, p_s$. We show that if $s > 1$, the number $s$ can be decreased. To this end, join any pair of exceptional points, e.g., $p_1$ and $p_2$, by a simple polygonal line $p_1p_2$ in $\Gamma$ and enclose this polygonal line in a very narrow closed polyhedral neighborhood $\bar{E}^n$, homeomorphic to a closed $n$-simplex (that is, embed the broken line in a narrow tube closed at both ends). Leaving the mapping $C$ unchanged in the exterior and on the boundary of the neighborhood $\bar{E}^n$, we suppress it at the points of $E^n$ and then extend it again to the set $\bar{E}^n \setminus p_2$. Then the redefined mapping $C$ has a smaller number of exceptional points.

By applying this construction as many times as necessary we arrive at a continuous mapping defined at all the points of $\bar{\Gamma}$ except for a single point $p_*$. This mapping is an extension of the originally prescribed mapping of $S_0^{n-1} \cup S_1^{n-1}$ into $S^{n-1}$. We must now remove the one remaining exceptional point $p_*$.

Suppose that $p_* \in T_*^n \in K$. It is no restriction to assume that $\bar{T}_*^n \subset \Gamma$. We orient the domain $\Gamma_0 = T_1^n \setminus (\bar{T}_0^n \cup \bar{T}_*^n)$ coherently with the prescribed orientation of $\Gamma$ (and $R^n$) and let $z_0^{n-1}$ be the boundary of the oriented domain $\Gamma_0$. It is clear that $z_0^{n-1} = z^{n-1} - \Delta t_*^n$, where $t_*^n$ is the orientation of $T_*^n$ coherent with that of $\Gamma$. The degree of the mapping $C$ on $z_0^{n-1}$, equal to the sum of its degrees on $z^{n-1}$ and $\Delta t_*^n$, is, in view of 2°, equal to zero. Since the degree of $C$ on $z^{n-1}$ is also equal to zero by assumption, the degree of $C$ on $\Delta t_*^n$ is also equal to zero. Therefore, retaining the mapping $C$ unchanged in the exterior and on the boundary of $T_*^n$ and suppressing it in the interior of $T_*^n$, we can use 6.23 to extend it again to the closed simplex $\bar{T}_*^n$. The final mapping is an extension of $C$ to all of $\bar{\Gamma}$.

This completes the proof of the extension theorem, and thus of Hopf's theorem.

## *Chapter XVII*

# FIXED POINTS OF CONTINUOUS MAPPINGS OF POLYHEDRA

This chapter is devoted to a proof of the celebrated Lefschetz-Hopf fixed point theorem. The theorem expresses the algebraic number of fixed points of a continuous mapping of a polyhedron into itself in terms of a homology type numerical invariant of the mapping, that is, in terms of a number determined by the endomorphisms of the Betti groups of the polyhedron induced by the mapping.

The Lefschetz-Hopf theorem was proved by Lefschetz in 1926 for manifolds and mappings with a finite number of fixed points. In 1927 Hopf gave a completely new and elementary proof, at the same time disposing of the more general case. The result was a theorem embracing almost all the facts known about the fixed points of polyhedra (and not only manifolds) and subsuming—even in the case of manifolds—as special cases the majority of previously known fixed point theorems (in particular, the theorems discussed in the preceding chapter).

The entire proof, which we give in this chapter, is due to Hopf. The chapter is itself a revision of Chapter 14, originally written by Hopf, of our joint work (Alexandroff-Hopf, *Topologie I*).

## §1. The fixed point theorem

**§1.1. Fixed simplexes.** Let $K_\beta$ be a subdivision of an $n$-dimensional triangulation $K_\alpha$ and let $S_\alpha^\beta$ be a simplicial mapping of $K_\beta$ into $K_\alpha$. If $T_\beta^r$ is an $r$-simplex of $K_\beta$, the dimension of the simplex $S_\alpha^\beta T_\beta^r$ does not exceed $r$; if it is equal to $r$, it may happen that $T_\beta^r$ is contained in $S_\alpha^\beta T_\beta^r$, that is, *$T_\beta^r$ may be a simplex of the subdivision of $S_\alpha^\beta T_\beta^r \in K_\alpha$ in $K_\beta$*. In that case $T_\beta^r$ is called a *fixed simplex* of $S_\alpha^\beta$.

If $t_\beta^r$ is an orientation of a fixed simplex $T_\beta^r$ (briefly: *$t_\beta^r$ is an oriented fixed simplex*), then $t_\alpha^r = S_\alpha^\beta t_\beta^r$ is also an oriented simplex and its subdivision $s_\beta^\alpha t_\alpha^r$ in $K_\beta$ is a chain; the coefficient of $t_\beta^r$ in this chain is

$$(s_\beta^\alpha S_\alpha^\beta t_\beta^r \cdot t_\beta^r) = \pm 1. \tag{1.10}$$

This coefficient is independent of the choice of the orientation $t_\beta^r$ of $T_\beta^r$, for if $t_\beta^r$ is replaced by $-t_\beta^r$, both terms in the scalar product (1.10) change sign. A fixed simplex $T_\beta^r \in K_\beta$ is called a *positive fixed simplex* if the scalar product (1.10) is $+1$, and a *negative fixed simplex* if it is $-1$.

We now introduce the following definition: *The algebraic number of fixed $r$-simplexes of a simplicial mapping $S_\alpha^\beta$ is the difference between the number of positive and the number of negative fixed $r$-simplexes of $S_\alpha^\beta$.*

We shall give another interpretation of this number.

Let $L_\beta^r = L_0^r(K_\beta)$ be the group of all integral $r$-chains of $K_\beta$. A simplicial mapping $S_\alpha^\beta$ of $K_\beta$ into $K_\alpha$ induces a homomorphism (denoted by the same letter; VII, 8.2) of $L_\beta^r$ into the group $L_\alpha^r = L_0^r(K_\alpha)$, and the subdivision of $K_\alpha$ (that is, the subdivision operator from $K_\alpha$ to $K_\beta$) induces an isomorphism $s_\beta^\alpha$ of $L_\alpha^r$ into $L_\beta^r$. Hence the mapping $s_\beta^\alpha S_\alpha^\beta$ is an endomorphism $f$ of $L_\beta^r$. We assert that the trace $Sp_f\, L_\beta^r$, or briefly, $Sp\, L_\beta^r$ of $f$ (Appendix 2, 2.7) is equal to the algebraic number of fixed $r$-simplexes of $S_\alpha^\beta$.

Indeed, the oriented $r$-simplexes $t_{\beta i}^r$ $(1 \leq i \leq \rho^r)$ of $K_\beta$ form a basis for $L_\beta^r$. The endomorphism $f$ is defined by the system of equations

$$ft_{\beta i}^r = \sum_{j=1}^{\rho^r} a_{ij} t_{\beta j}^r \qquad (1 \leq i \leq \rho^r), \tag{1.11}$$

where the coefficients $a_{ij}$ are 0, $+1$ or $-1$. A simplex $|\, t_{\beta i}^r \,|$ is fixed if, and only if, $a_{ii} \neq 0$; it is a positive or negative fixed simplex if $a_{ii}$ is $+1$ or $-1$, respectively. Consequently, the algebraic number of fixed $r$-simplexes is equal to

$$\sum_{i=1}^{\rho^r} a_{ii} = Sp\, L_\beta^r. \tag{1.1}$$

§**1.2. Hopf's formula.** We have shown that a sufficient condition for the existence of fixed simplexes of $S_\alpha^\beta$ is the nonvanishing of the trace $Sp\, L_\beta^r$ of the endomorphism $f = s_\beta^\alpha S_\alpha^\beta$ of the group $L_\beta^r$. The existence of fixed simplexes (as well as their number) is a property of the simplicial mapping $S_\alpha^\beta$. We shall show, however, that it is a homology property of the mapping $\bar{S}_\alpha^\beta$ of the polyhedron $\|\, K_\alpha \,\|$ into itself induced by the simplicial mapping $S_\alpha^\beta$, and shall clarify the topological meaning of this property. To this end, we express the algebraic number of fixed simplexes in terms of a number characterizing the behavior of the Betti groups under the induced endomorphisms; the result is the formula discovered by Hopf and named after him. Hopf's formula is proved by methods analogous to those used in the proof of the Euler-Poincaré formula, with only this essential difference, that the ranks of the various groups are now replaced by the traces of the endomorphisms to which these groups are subjected.

Because the operators $\Delta$ and $s_\beta^\alpha S_\alpha^\beta$ commute, the endomorphism $f = s_\beta^\alpha S_\alpha^\beta$ of $L_\beta^r$ induces endomorphisms of the subgroups $Z_\beta^r = Z_\Delta^r(K_\beta, J)$, $H_\beta^r = H_\Delta^r(K_\beta, J)$ and $\hat{H}_\beta^r = \hat{H}_\Delta^r(K_\beta, J)$. It follows that $f$ induces endomorphisms of the factor groups $L_\beta^r / Z_\beta^r = A_\beta^r$ and $Z_\beta^r / \hat{H}_\beta^r = \Delta_{00\beta}^r$, which we also denote by $f$. The factor groups $A_\beta^r$ and $\Delta_{00\beta}^r$, like the groups $Z_\beta^r$, $H_\beta^r$ and $\hat{H}_\beta^r$, are free Abelian groups. Hence the traces $Sp\, Z_\beta^r$, $Sp\, H_\beta^r$, $Sp\, \hat{H}_\beta^r$, $Sp\, A_\beta^r$, $Sp\, \Delta_{00\beta}^r$ $(0 \leq r \leq n)$ are all defined.

Since the reasoning in the sequel involves only $K_\beta$, we shall omit the index $\beta$.

We shall prove that

$$(1.21) \qquad Sp\,A^r = Sp\,H^{r-1}.$$

The operator $\Delta$ is a homomorphism of $L^r$ onto $H^{r-1}$ and $\Delta f x^r = f \Delta x^r$ for every $x^r \in L^r$. The kernel of this homomorphism is $Z^r$. It is clear that the isomorphism $\Delta$ of $A^r = L^r/Z^r$ onto $H^{r-1}$ induced by the homomorphism $\Delta$ of $L^r$ onto $H^{r-1}$ also satisfies the relation

$$\Delta f = f\Delta;$$

hence the isomorphism $\Delta$ of $A^r$ onto $H^{r-1}$ transforms the endomorphism $f$ of $A^r$ into the endomorphism $f$ of $H^{r-1}$ and (1.21) follows.

In view of the additivity of the traces (Appendix 2, Theorem 2.8), we have

$$Sp\,A^r = Sp\,(L^r/Z^r) = Sp\,L^r - Sp\,Z^r,$$

and (1.21) can be rewritten as

$$(1.22) \qquad Sp\,L^r - Sp\,Z^r = Sp\,H^{r-1}.$$

Furthermore, since $\Delta_{00}{}^r = Z^r/\hat{H}^r$, where $\hat{H}^r$ is the division closure of $H^r$ in $Z^r$, we have (since $Sp\,\hat{H}^r = Sp\,H^r$; see Appendix 2, 2.8, Remark)

$$(1.23) \qquad Sp\,Z^r - Sp\,H^r = Sp\,\Delta_{00}{}^r.$$

Adding (1.23) and (1.22), we obtain

$$Sp\,L^r - Sp\,H^r = Sp\,H^{r-1} + Sp\,\Delta_{00}{}^r,$$

or

$$(1.24) \qquad Sp\,L^r = Sp\,\Delta_{00}{}^r + Sp\,H^r + Sp\,H^{r-1}.$$

Multiplying (1.24) by $(-1)^r$, summing over $r$ from 0 to $n$ and recalling that $H^{-1} = H^n = 0$ (so that $Sp\,H^{-1} = Sp\,H^n = 0$), we obtain *Hopf's formula*:

$$(1.25) \qquad \sum_{i=0}^{n} (-1)^r Sp\,L^r = \sum_{r=0}^{n} (-1)^r Sp\,\Delta_{00}{}^r.$$

Hopf's formula is a generalization of the Euler-Poincaré formula since, if $K_\beta = K_\alpha$ and the simplicial mapping $S_\alpha{}^\beta$ is the identity (as is also $s_\beta{}^\alpha$), then all the endomorphisms $f$ are identity mappings of the corresponding groups and their traces are equal to the ranks of the corresponding groups. In particular, $Sp\,L^r = \rho^r$ and $Sp\,\Delta_{00}{}^r = \pi^r$.

An immediate consequence of (1.25) is the following

THEOREM 1.2. (EXISTENCE OF FIXED SIMPLEXES.) *If*

$$\sum_{r=0}^{n} (-1)^r\, Sp\,\Delta_{00}{}^r \neq 0,$$

*then there exists at least one fixed simplex.*

Indeed, in this case, there is an $r$ such that $Sp\, L^r \neq 0$. Then for this $r$ there exists an $i$ such that $a_{ii} \neq 0$ in (1.1) and (1.11); consequently, $| t_i^r |$ is a fixed simplex.

For instance, let $K_\beta$ be a triangulation of a closed $n$-simplex. Then $\Delta_{00}^r = 0$ for $r > 0$; therefore, $Sp\, \Delta_{00}^r = 0$ for $r > 0$. But $Sp\, \Delta_{00}^0 = 1$ (as it is for every connected complex). Consequently, $\sum_{r=0}^n Sp\, \Delta_{00}^r = 1$ and $K_\beta$ contains at least one fixed simplex.

**§1.3. The Lefschetz number of a continuous mapping of a polyhedron into itself.** Let $C$ be a continuous mapping of an $n$-dimensional polyhedron $P$ into itself. According to XI, 3 and 4, $C$ induces a homomorphism $C$ of $\Delta_0^r(P)$ into itself; this homomorphism also maps the torsion group $\Theta^r(P) \subseteq \Delta_0^r(P)$ into itself. Therefore, the homomorphism $C$ of $\Delta_0^r(P)$ into itself induces an endomorphism of $\Delta_{00}^r \approx \Delta_0^r(P)/\Theta^r(P)$, which we again denote by $C$. Since $\Delta_{00}^r$ is a free Abelian group, the trace $Sp\, \Delta_{00}^r = Sp_C\, \Delta_{00}^r$ is defined. If $P = \| K_\alpha \| = \| K_\beta \|$ and $C = S_\alpha^\beta$ (where $K_\alpha$, $K_\beta$ and $S_\alpha^\beta$ have the same meaning as in the preceding subsections), then $Sp_C\, \Delta_{00}^r$ coincides with the trace $Sp_f\, \Delta_{00}^r$ defined in 1.2 (since every cycle of $K_\alpha$ is in the same homology class of $P$ as its subdivision in $K_\beta$).

DEFINITION 1.3. *The Lefschetz number of a continuous mapping $C$ of a polyhedron $P$ into itself is*

$$\Lambda_C = \sum_{r=0}^n (-1)^r\, Sp\, \Delta_{00}^r.$$

REMARK. The number $\Lambda_C$ expresses a property of the homomorphism $C$ of $\Delta_{00}^r$ into itself, i.e., a property of the homology type of the mapping $C$. Therefore, every two completely homologous and so, a fortiori, every two homotopic mappings of a polyhedron $P$ into itself have the same Lefschetz number.

FIXED POINT THEOREM 1.3. *If $\Lambda_C \neq 0$, then $C$ has at least one fixed point, i.e., there is a point $p \in P$ such that $Cp = p$.*

*Proof.* Suppose $\Lambda_C \neq 0$. Define on $P$ the real-valued function $\varphi(p) = \rho(p, Cp)$. We must show that $\varphi(p)$ vanishes for at least one point of $P$. Since $P$ is compact and $\varphi$ is continuous, it is sufficient to show that for every $\epsilon > 0$ there exists a point $p \in P$ for which $\varphi(p) = \rho(Cp, p) < \epsilon$.

Suppose $\epsilon > 0$ is prescribed. We approximate $C$ by a simplicial mapping $S_\alpha^\beta$ of a subdivision $K_\beta$ of a triangulation $K_\alpha$ of $P$. We assume that the mesh of $K_\alpha$ is so small that the following conditions are satisfied (see XI, 5):

1°) $C$ is homotopic to $\tilde{S}_\alpha^\beta$.

2°) The mesh of $K_\alpha$ is $< \epsilon/2$.

3°) $\rho(Cp, \tilde{S}_\alpha^\beta p) < \epsilon/2$ for all $p \in P$.

By 1°), $\Lambda_{\tilde{S}_\alpha^\beta} = \Lambda_C$ (see the Remark after Def. 1.3).

Hence $\Lambda_{\tilde{S}_\alpha^\beta} \neq 0$ and $S_\alpha^\beta$ has a fixed simplex according to Theorem 1.2.

If $p$ is an arbitrary point of a fixed simplex $T_\beta \in K_\beta$, then $p$ and $\tilde{S}_\alpha^\beta p$ are contained in the same simplex $S_\alpha^\beta T_\beta \in K_\alpha$; hence, according to 2°),

$$\rho(p, \tilde{S}_\alpha^\beta p) < \epsilon/2. \tag{1.31}$$

But, by 3°),

$$\rho(Cp, \tilde{S}_\alpha^\beta p) < \epsilon/2. \tag{1.32}$$

It follows from (1.31) and (1.32) that

$$\rho(p, Cp) < \epsilon.$$

This completes the proof.

**§1.4. Examples.** 1°) It follows immediately from the geometric meaning of the 0th Betti group that $Sp\ \Delta_{00}^0 = 1$ for every mapping of a connected polyhedron $P$. If the Betti numbers $\pi^r(P)$ are equal to zero for all $r > 0$, i.e., $\Delta_{00}^r = 0$ for $r > 0$, then all the traces $Sp\ \Delta_{00}^r$ are also equal to zero for $r > 0$. In that case, $\Lambda_C = 1$ for every mapping $C$ of $P$ into itself, and we have

1.41. *Every continuous mapping of a connected polyhedron $P$ into itself, all of whose Betti numbers (except $\pi^0$) are equal to zero, has at least one fixed point.*

Examples of such polyhedra are the closed $n$-simplexes and the projective spaces of even dimension.

2°) In the following two theorems we utilize the fact that homotopic mappings are homologous and that consequently the Lefschetz numbers of such mappings are equal.

If a polyhedron $P$ is mapped into a single point, $Sp\ \Delta_{00}^0 = 1$ and $Sp\ \Delta_{00}^r = 0$ for $r > 0$, so that $\Lambda_C = 1$. Hence,

1.42. *If a continuous mapping $C$ of a polyhedron $P$ into itself is homotopic to zero, then $C$ has at least one fixed point.*

3°) The identity mapping of a polyhedron $P$ induces identity isomorphisms of the Betti groups. Therefore, $Sp\ \Delta_{00}^r = \pi^r$ for every $r$, and the Lefschetz number is equal to $\sum_{r=0}^{n} (-1)^r \pi^r$, that is, the Euler characteristic of $P$. Consequently,

1.43. *If the Euler characteristic of a polyheron $\| K \|$ is different from zero and a continuous mapping $C$ of $\| K \|$ into itself is homotopic to the identity, then $C$ has a fixed point.*

All surfaces (closed and with boundary), except for the torus, the Klein bottle, a circular ring and the Möbius band, satisfy Theorem 1.43.

4°) If $P$ is an $n$-dimensional closed orientable pseudomanifold whose Betti numbers of dimensions $1, 2, \cdots, n - 1$ are equal to zero (for instance, if $P$ is an $n$-sphere or projective $2n$-space) and $C$ is a mapping of degree $\gamma$ of $P$ into itself, then $Sp\, \Delta_{00}^0 = 1$, $Sp\, \Delta_{00}^n = \gamma$ and $Sp\, \Delta_{00}^r = 0$ for $0 < r < n$. Therefore, $\Lambda_C = 1 + (-1)^n\gamma$.

It follows, in particular, that:

1.44. *If all the Betti numbers of dimensions* $1, 2, \cdots, n - 1$ *of a closed orientable n-dimensional pseudomanifold* $P$ *are equal to zero* [for examples see the parenthetical remark in the first paragraph of 4°)], *then every mapping of* $P$ *into itself of degree different from* $(-1)^{n+1}$ *has a fixed point.*

**§1.5. Some remarks on the Lefschetz number.** We saw that the condition $\Lambda_C \neq 0$ is sufficient for the existence of fixed points of a mapping $C$. We show that this condition is not necessary. To this end, we prove the following very elementary proposition:

1.5. *It is possible to construct a continuous mapping* $C_0$ *of a lemniscate onto itself, whose Lefschetz number is zero, but which has a fixed point. (Every mapping* $C$ *homotopic, or even* $\mathfrak{N}$*-homologous, to* $C_0$ *has the same property.)*

The proof is preceded by the following lemma:

1.51. *If* $C$ *is a mapping of a circumference into itself of degree* $c$, *then* $C$ *has at least* $|c - 1|$ *fixed points.*

In particular, the lemma implies that every mapping of a circumference into itself which has no fixed points has degree $+1$ and is, therefore, homotopic to the identity.

*The proof of Lemma* 1.51 is remarkably simple. We consider the angle function $F(x)$ corresponding to the mapping $C$ (see II, proof of Theorem 2.51). Then $F(1) - F(0) = 2\pi c$. Setting $f(x) = F(x) - 2\pi x$, we have $f(1) - f(0) = 2\pi(c - 1)$; whence it follows that $f(x)$ is an integral multiple of $2\pi$ for at least $|c - 1|$ values of $x$. Denoting these values of $x$ by $x_i$, we have $F(x_i) = 2\pi x_i + 2\pi k_i$, $k_i$ an integer. Then the points of the circumference corresponding to the points $x_i$ are fixed points of $C$, which proves Lemma 1.51.

*Proof of Theorem* 1.5. Instead of a lemniscate, we consider the union of two circumferences $\bar{z}_1$ and $\bar{z}_2$ tangent to each other at a point $p_0$. Denote by $z_1$ and $z_2$ arbitrary orientations of $\bar{z}_1$ and $\bar{z}_2$. On each of these circumferences we shall measure angles from $p_0$ in the sense corresponding to the prescribed orientations of the circumferences. We then define the mapping $C_0$ by the formulas

$$C_0(\alpha_1) = -\alpha_1 \quad \text{on } \bar{z}_1,$$

$$C_0(\alpha_2) = 2\alpha_2 \quad \text{on } \bar{z}_2,$$

where $\alpha_i$ $(i = 1, 2)$ denotes the angular coordinate we have just introduced on $\bar{z}_i$.

The mapping $C_0$ (and every mapping $\mathfrak{N}$-homologous to $C_0$) maps the cycles $z_1$ and $z_2$ into $-z_1$ and $2z_2$, respectively, so that $Sp\, \Delta_{00}^1 = 1$ and consequently $\Lambda = 0$.

On the other hand, Lemma 1.51 implies that $C_0$, which has degree $-1$ on $\bar{z}_1$, has at least $|-1 - 1| = 2$ fixed points on $\bar{z}_1$.

We shall now prove that every continuous mapping $C$, which is $\mathfrak{N}$-homologous to $C_0$, has at least one fixed point on $\bar{z}_1$ (in spite of the fact that $C$ may not map all of $\bar{z}_1$ into itself). With the parenthetical remark in mind, we construct an auxiliary mapping $C'$ of $\bar{z}_1$ into itself as follows: If $p \in \bar{z}_1$ implies $C(p) \in \bar{z}_1$, set $C'(p) = C(p)$. If $C(p) \in \bar{z}_2$, set $C'(p) = p_0$. The mapping $C'$ is continuous and has the following property: If $p$ is a point of $\bar{z}_1$ different from $p_0$ and $C_1$, $C_1'$ are sufficiently good simplicial approximations to $C$ and $C'$ (restricted to $\bar{z}_1$), then $C_1(z_1)$ and $C_1'(z_1)$ cover $p$ the same (algebraic) number of times. (This easily follows from the fact that $C$ and $C'$ coincide at all the points of $\bar{z}_1$ mapped by $C$ into $\bar{z}_1$.) But since $C(z_1) = -z_1$, this number is $-1$ for $C_1$, so that it is also $-1$ for $C_1'$. Hence the mapping $C'$ of $\bar{z}_1$ into itself has degree $-1$ and therefore has at least two fixed points. If $p$ is a fixed point of $C'$ different from $p_0$, then $C'(p) = p \neq p_0$. Hence $C'(p) = C(p)$, and so $C(p) = p$.

## §2. The index of a fixed point

**§2.1. Definition of the index.** Let $C$ be a continuous mapping of a domain $\Gamma \subseteq R^n$ into $R^n$. We shall call the vector $v^C(x) = \overrightarrow{xC(x)}$ the *displacement vector at the point $x$ induced by the mapping $C$*. The vectors $v^C(x)$ form a vector field $\mathfrak{V}^C$ defined at all the points of $\Gamma$; $\mathfrak{V}^C$ is referred to as the *displacement field of $C$*. The zeros of this field coincide with the fixed points of $C$.

Let $p$ be an isolated fixed point of $C$, that is, an isolated zero of the field $\mathfrak{V}^C$. The index of this zero (see XVI, 5.2) is called the *index of the fixed point $p$*. In view of the discussion of XVI, 5.2, this index may be determined, for instance, as follows:

2.11. We suppose $R^n$ is given a fixed orientation and that all its $(n-1)$-spheres are oriented coherently with the chosen orientation of $R^n$. Let $\bar{E}^n$ be a closed solid sphere in $\Gamma$, bounded by the sphere $S^{n-1} = \bar{E}^n \setminus E^n$, and containing no fixed point of $C$ different from $p$. We now take a sphere of directions (that is, a fixed sphere, e.g., the unit sphere with center at the origin and radius 1) $S_0^{n-1}$ of $R^n$ and assign to each point $x \in S^{n-1}$ that point $C_0x$ of $S_0^{n-1}$ at which $S_0^{n-1}$ intersects the ray from its center parallel to the displacement vector $v^C(x)$. The degree of the mapping $C_0$ of $S^{n-1}$ into $S_0^{n-1}$ is also called *the index of the fixed point $p$* and is denoted by $\mathrm{ind}_C\, p$.

REMARK. The index $\mathrm{ind}_C\, p$ is independent of the choice of the sphere $S^{n-1}$ used to define it; moreover, the sphere $S^{n-1}$ can be replaced by an arbitrary proper cycle $\mathfrak{z}^{n-1}$, relative to which the point $p$ has index 1 (see XVI, 5.1, 5.2). This is a consequence of the fact that $\mathrm{ind}_C\, p$ is the index $\mathfrak{v}(C_0\mathfrak{z}^{n-1}, o)$ and of the elementary properties of the index.

DEFINITION 2.12. If a domain $\Gamma$ contains only a finite number of fixed

**§2.3. Normal fixed points. Topological invariance of the index of a fixed point.** Let $\Gamma$ and $C$ have the same meaning as in 2.1. Suppose that a fixed point $o \in \Gamma$ (of $C$) has the following property:

*There exists a closed n-cell (that is, a topological image of a closed n-simplex)* $\bar{E}^n = E^n \cup S^{n-1}$ *such that o is the only fixed point of C in* $E^n$ *and such that* $C(S^{n-1}) \cap \bar{E}^n = 0$.

Then we say that $o$ is a *normal* fixed point.

We orient the cell $E^n$ coherently with the prescribed orientation of $\Gamma$. The boundary of the oriented cell $E^n$ is a proper (singular) cycle $\mathfrak{z}^{n-1}$. We shall prove the following proposition:

2.31. *With the notation as above, the index of a normal fixed point o is equal to the degree of C in the point o of* $\bar{E}^n$, *or, equivalently, to the index of o relative to the cycle* $C\mathfrak{z}^{n-1}$.

*Proof.* By means of a topological mapping of a closed $n$-simplex $\bar{T}^n$ onto $\bar{E}^n$, transfer the affine geometry of $\bar{T}^n$ to $\bar{E}^n$. Using this geometry, denote by $p_\theta$, for $0 \leq \theta \leq 1$ and $p \in S^{n-1}$, the point dividing the "segment" $[po]$ in the ratio $\theta:(1 - \theta)$. Since $o$ is a normal fixed point, $p_\theta$ cannot be contained in $C(S^{n-1})$ for $\theta \geq 0$. Hence $p_\theta \neq C(p)$.

The deformation

$$v_\theta(p) = p_\theta C(p)$$

is a continuous transformation of the field $\mathfrak{V}^C = \mathfrak{V}_0$ into the field $\mathfrak{V}_1$ consisting of the vectors

$$v_1(p) = oC(p),$$

and the vectors of the field do not vanish during the deformation. It follows that the fields $\mathfrak{V}_0$ and $\mathfrak{V}_1$ have the same characteristic on $\mathfrak{z}^{n-1}$. But the characteristic of $\mathfrak{V}_1$ on $\mathfrak{z}^{n-1}$ is the index of $o$ relative to $C(\mathfrak{z}^{n-1})$, which was to be proved.

The importance of the concept of normal fixed points depends to a great extent on the following propositions, which together with Theorem 2.31, enable us to prove the invariance of the index of a fixed point.

2.32. An isolated fixed point $o \in \Gamma$ of a mapping $C$ of a domain $\Gamma \subseteq R^n$ into $R^n$ is said to be *normalizable* if there exists a closed $n$-cell $\bar{E}^n \subset \Gamma$ containing $o$ in its interior and a deformation $C_\theta$, $0 \leq \theta \leq 1$, of $C_0 = C$, restricted to $\bar{E}^n$, satisfying the following conditions:

1) No mapping $C_\theta$, $0 \leq \theta \leq 1$, has a fixed point on the boundary of $E^n$.

2) The point $o$ is a normal fixed point of $C_1$.

2.33. *Every isolated fixed point* $o \in \Gamma$ *of a continuous mapping C of a domain* $\Gamma$ *is normalizable.*

*Proof.* Let $T^n$, $o \in T^n$, $\bar{T}^n \subset \Gamma$ be any $n$-simplex such that $\bar{T}^n$ contains no fixed point of $C$ different from $o$. Denote by $\varphi$ an arbitrary nonnegative continuous function defined on all of $\bar{T}^n$, assuming the value 0 at $o$ and the value 1 at all points of the boundary of $\bar{T}^n$. Let $d$ be the diameter of $T^n$. Now define a deformation $C_\theta$ of the mapping $C = C_0$ of $\bar{T}^n$ as follows:

For every $x \in \bar{T}^n$ denote by $C_\theta x$ the point of the ray $xC(x)$ at a distance $\rho[C(x), x] + \theta d\varphi(x)$ from $x$. No mapping $C_\theta$ has a fixed point on the boundary of $T^n$; the point $o \in T^n$ is the only fixed point of $C_1$ in $\bar{T}^n$. It is normal, because

$$\rho[C_1(x), x] = \rho[C(x), x] + d > d$$

for $x \in \bar{T}^n \setminus T^n$, so that $C(x) \notin \bar{T}^n$.

We apply Theorem 2.33 to prove

THEOREM 2.34. THE TOPOLOGICAL INVARIANCE OF THE INDEX OF AN ISOLATED FIXED POINT OF A CONTINUOUS MAPPING. *Let $o$ be an isolated fixed point of a mapping $C$ of a domain $\Gamma \subseteq R^n$ into $R^n$. If $F$ is a topological mapping of $R^n$ onto itself (or onto another Euclidean space $R_1^n$), then the index of the point $o' = F(o)$ with respect to the mapping $FCF^{-1}$ is equal to the index of the fixed point $o$ with respect to $C$.*

The same theorem can obviously be formulated as

2.340. Let $R$ be a topological space homeomorphic to $R^n$. Let $\Gamma$ be a domain of $R$ and let $C$ be a continuous mapping of $\Gamma$ into $R$ having an isolated fixed point $o$. Suppose that $F$ is a topological mapping of $R^n$ onto $R$ and transfer the Euclidean geometry of $R^n$ to $R$ by means of this mapping. Then the index of $o$ relative to $C$, defined as above by means of the induced Euclidean geometry in $R$, is independent of this geometry, i.e., it is the same for all topological mappings $F$ of $R^n$ onto $R$.

If $o$ is a normal fixed point, Theorem 2.34 follows from 2.31 and the topological invariance of the local degree of a continuous mapping (XVI, Theorem 4.22).

In the general case Theorem 2.34 follows from 2.33. However, it is more convenient to prove 2.340.

We note first that the concept of normal fixed point is independent of the geometry introduced in $R$ and that we have already proved that every normal fixed point of a given continuous mapping $C$ has the same index in all geometries.

Now let $C$ be a continuous mapping of a domain $\Gamma$ into $R$, with $o \in \Gamma$ as an isolated fixed point. We consider a cell $\bar{E}^n \subset \Gamma$ containing $o$ in its interior and a continuous deformation $C_\theta$ of $C_0 = C$ into a mapping $C_1$ with $o$ as a normal fixed point.

We suppose that the deformation $C_\theta$ satisfies 1) and 2) of Def. 2.32. It follows from these conditions that the characteristic of the vector fields $\mathfrak{B}^{C_0}$ and $\mathfrak{B}^{C_1}$ is the same on the boundary of $\bar{E}^n$ no matter which geometry is introduced in $R$. Hence, $\text{ind}_{C_0}\, o = \text{ind}_{C_1}\, o$ in every geometry. But $\text{ind}_{C_1}\, o$ is the same in all geometries; consequently, $\text{ind}_{C_0}\, o$ is the same in all geometries.

**§2.4. Fixed points of affine mappings.** We give still another application of normal fixed points: a definition of the index of a normal fixed point of an affine mapping.

THEOREM 2.4. *Let* $\bar{T}^n$ *be a closed n-simplex in* $R^n$; *denote by* $\tilde{S}$ *an affine mapping of* $\bar{T}^n$ *into* $R^n$ *such that* $\bar{T}^n \subseteq \tilde{S}\bar{T}^n$ *and such that* $\tilde{S}$ *has no fixed points on* $\bar{T}^n \setminus T^n$. *Then* $T^n$ *contains precisely one fixed point with index* $+1$ $(-1)$ *if* $\tilde{S}$ *is positive (negative).*

*Proof.* Since $\bar{T}^n \subseteq \tilde{S}\bar{T}^n = \bar{T}_1^n$, the dimension of $T_1^n$ is equal to $n$ and, consequently, $\tilde{S}$ is non-singular, that is, the inverse mapping $\tilde{S}^{-1}$ exists. Since $\bar{T}^n \subseteq \bar{T}_1^n$, the mapping $\tilde{S}^{-1}$ of $\bar{T}_1^n$ onto $\bar{T}^n$ has at least one fixed point (see 1.41). Such a fixed point is also a fixed point for $\tilde{S}$. Since $\tilde{S}$ has no fixed points in $\bar{T}^n \setminus T^n$, the fixed point must be in $T^n$.

If $\tilde{S}$ had two fixed points $o_1$ and $o_2$ in $T^n$, all the points of the straight line $o_1o_2$, which is contained in $\bar{T}^n$, would be fixed points, since $\tilde{S}$ is affine; in particular, there would be fixed points on the intersection of this straight line with the boundary of $T^n$, contrary to our hypothesis. Hence, $T^n$ contains only one fixed point. This point is a normal fixed point and, consequently, by 2.3 its index is equal to the degree of $\tilde{S}$ in the point $o$, i.e., $+1$ or $-1$ depending on the sign of $\tilde{S}$. We shall apply this theorem in 3.4.

## §3. The algebraic number of fixed points of a continuous mapping of a polyhedron into itself

**§3.1. Definition of regular fixed points. Formulation of the fundamental theorem.** Let $P$ be a pure (see VI, Def. 5.24) $n$-dimensional polyhedron, let $C$ be a continuous mapping of $P$ into itself and let $o$ be a fixed point of $C$. We call the fixed point $o$ *regular* (with respect to a triangulation $K$ of $P$) if it is an isolated fixed point (that is, if it has a neighborhood containing no other fixed points) and is contained in an $n$-simplex $T$ of the triangulation $K$. Because $C$ is continuous, the point $o$ has a neighborhood $U$ whose image $CU$ is also contained in $T$. Consequently, if we regard $T$ as part of the Euclidean space $R^n$ which carries $T$, we can speak of the index of the fixed point $o$.

If $C$ has only regular fixed points (relative to $K$), the sum of their indexes is called the *algebraic number of fixed points of* $C$. The following theorem asserts that the algebraic number of fixed points of $C$ is completely determined by the homology type of $C$:

THEOREM 3.1. *If a continuous mapping* $C$ *of the body* $P = \|K\|$ *of a pure n-complex* $K$ *onto itself has only regular fixed points (relative to* $K$*), then the algebraic number of fixed points is equal to* $(-1)^n\Lambda_C$.

We shall prove this theorem in 3.3, 3.4 and 3.5. It is a stronger form of the fixed point Theorem 1.3. The theorems of 1.4 can now also be strengthened. If $P$ is connected and $\pi^r(P) = 0$ for $r > 0$, the algebraic number of fixed points of an arbitrary continuous mapping of $P$ (all of whose fixed points are regular) is equal to $(-1)^n$; if $C \simeq 0$, then (for arbitrary $P = \| K \|$) its algebraic number of fixed points is equal to the Euler characteristic of $K$ multiplied by $(-1)^n$; if $C$ is a mapping of $S^n$ into itself of degree $d$, the algebraic number of fixed points of $C$ is equal to $(-1)^n + d$; etc.

§3.2. **Generalization of Theorem 3.1.** Theorem 3.1 can be extended to an arbitrary continuous mapping (without the restriction that its fixed points be regular), that is, to a continuous mapping whose algebraic number of fixed points is not determined in advance. This can be done by means of the following

THEOREM 3.2. *Let $C$ be an arbitrary continuous mapping of a pure $n$-dimensional polyhedron $P$ into itself and let $\epsilon$ be an arbitrarily small positive number. There exists a continuous mapping $C_\epsilon$ of $P$ into itself and a triangulation $K$ of $P$ with the following properties:*

1) *The mappings $C$ and $C_\epsilon$ differ from each other by less than $\epsilon$:*

$$\rho[C(x), C_\epsilon(x)] < \epsilon$$

*for all $x \in P$.*

2) *All the fixed points of $C_\epsilon$ are regular with respect to $K$.*

*In addition, if $\epsilon$ is sufficiently small, all the mappings $C_\epsilon$ satisfying 1) and 2) are homotopic to $C$ and have the same algebraic number of fixed points, equal to $(-1)^n \Lambda_C$.*

The second part of Theorem 3.2 follows immediately from Theorem 3.1: if $\epsilon$ is chosen so small that the approximating mappings are all of the same homology type, the Lefschetz number of each of these mappings is equal to $\Lambda_C$; hence, by Theorem 3.1 the algebraic number of fixed points of each mapping is equal to $(-1)^n \Lambda_C$.

§3.3. **Regular fixed points of simplicial mappings.** Hence, we need prove only Theorem 3.1 and the first part of Theorem 3.2 (the existence of the mappings $C_\epsilon$).

We can strengthen Theorem 3.2 still further by requiring that the approximating mappings be *simplicial*. However, before formulating this strengthened version, we wish to characterize combinatorially the property that a simplicial mapping have only regular points.

3.30. Let $K_2$ be a subdivision of a pure $n$-dimensional triangulation $K_1$, and let $S$ be a simplicial mapping of $K_2$ into $K_1$. If $S$ has a fixed $r$-simplex $(r < n)$, then $\tilde{S}$ has at least one nonregular fixed point (relative to $K_1$). Conversely, if the only fixed simplexes of $S$ are $n$-simplexes, then $\tilde{S}$ has only regular fixed points (relative to $K_1$); furthermore, $\tilde{S}$ has precisely one fixed point in each fixed $n$-simplex of $K_2$.

*Proof.* If $T_2^r \in K_2$ is a fixed $r$-simplex of $S$, $\tilde{S}$ has at least one fixed point

$p \in T_2^r$. Since $p$ is in the simplex $T_1^r = ST_2^r$, it is not regular if $r < n$. Conversely, suppose $S$ has no fixed simplexes of dimension less than $n$; let $p$ be a fixed point of $\tilde{S}$ and let $T_2$ be its carrier in $K_2$ (i.e., the simplex of $K_2$ containing $p$). Then $T_2$ is a fixed simplex of $S$; hence its dimension is $n$. Its boundary contains no fixed points, since it consists of simplexes of dimension less than $n$. Hence, according to Theorem 2.4, $p$ is the unique fixed point in $T_2$. It is, therefore, a regular fixed point.

The stronger form of the first part of Theorem 3.2 alluded to above can be stated as

THE APPROXIMATION THEOREM 3.3. *For prescribed* $\epsilon > 0$ *and continuous mapping* $C$ *of a pure n-dimensional polyhedron* $P$ *into itself, there exist triangulations* $K_1$ *and* $K_2$ *of* $P$, *with* $K_2$ *a subdivision of* $K_1$, *and there exists a simplicial mapping* $S$ *of* $K_2$ *into* $K_1$ *satisfying the following conditions:*

1) *The inequality*

$$\rho[C(x), \tilde{S}(x)] < \epsilon$$

*is satisfied for all* $x \in P$.

2) *All the fixed simplexes* $T_2 \in K_2$ *of* $S$ *have dimension* $n$.

**§3.4. Reduction of Theorem 3.1 to the approximation theorem.** In addition to Theorem 3.3, we must still prove Theorem 3.1. However, we shall now show that Theorem 3.1 follows from Theorem 3.3.

Suppose that the continuous mapping $C$ satisfies the hypotheses of Theorem 3.1, and let $o_i$ $(1 \leq i \leq s)$ be its fixed points (there are only a finite number of these). Furthermore, let $T_i^n$ $(1 \leq i \leq s)$ be the carrier of $o_i$ in $K$ and let $T_{1i}^n$ be an $n$-simplex contained in $T_i^n$, containing $o_i$ and so small that $CT_{1i}^n \subset T_i^n$. The function $f(x) = \rho(x, Cx)$ has an infimum $b_i > 0$ on the set $P \setminus \bigcup_{i=1}^n T_{1i}^n$. In addition, according to Theorem 2.22, there exists for each $i$ a number $a_i > 0$ [the infimum of $f(x)$ on $\| \dot{T}_{1i}^n \|$] such that every mapping $C_1$ for which $\rho(C, C_1) < a_i$ has the same algebraic number of fixed points in $T_{1i}^n$ as $C$.

Suppose that $\epsilon$ is a positive number smaller than all of the numbers $a_i$ and $b_i$ $(1 \leq i \leq s)$, and let $S$ be a mapping satisfying 1) and 2) of Theorem 3.3.

Since $\epsilon < b_i$, $\tilde{S}$ has no fixed points outside the simplexes $T_{1i}^n$; since $\epsilon < a_i$, $C$ and $\tilde{S}$ have the same algebraic number of fixed points in each of the simplexes $T_{1i}^n$. Hence, if we denote by $J_C$ and $J_{\tilde{S}}$ the algebraic number of fixed points of $C$ and $\tilde{S}$, respectively, we have

$$J_C = J_{\tilde{S}}. \tag{3.41}$$

We may further suppose that $\epsilon$ is so small that $C$ and $\tilde{S}$ are homotopic and are, consequently, also homologous; then, in particular,

$$\Lambda_C = \Lambda_{\tilde{S}}. \tag{3.42}$$

Formulas (3.41) and (3.42) reduce the proof of the assertion

$$J_C = (-1)^n \Lambda_C$$

of Theorem 3.1 to the proof of the analogous assertion

$$J_{\tilde{S}} = (-1)^n \Lambda_{\tilde{S}}$$

for a simplicial mapping $S$ whose only fixed simplexes are $n$-simplexes. But for such a mapping (using the notation of §1) $Sp\ L^r = 0$ for $r < n$, and (1.25) becomes

$$(-1)^n\ Sp\ L^n = \Lambda_{\tilde{S}},$$

or

$$Sp\ L^n = (-1)^n \Lambda_{\tilde{S}}.$$

It was shown at the end of 1.1 that $Sp\ L^n$ is equal to the algebraic number of fixed $n$-simplexes, that is, the difference between the number of positive and negative fixed $n$-simplexes. Therefore, by Theorem 2.4, $Sp\ L^n$ is the sum of the indexes of the fixed points in the fixed $n$-simplexes. Since there are no other fixed points

$$J_{\tilde{S}} = (-1)^n \Lambda_{\tilde{S}}.$$

This completes the proof.

Our only remaining task is to prove the approximation Theorem 3.3.

**§3.5. Proof of the approximation theorem.** In this subsection $K_1$ will denote the barycentric subdivision of a triangulation $K$ and "a subdivision of a triangulation" will always mean a barycentric subdivision of some order.

Since every continuous mapping can be approximated with arbitrary accuracy by a simplicial mapping, we may suppose to begin with that $C = \tilde{S}_0$, where $S_0$ is a simplicial mapping of a subdivision $K^{(0)}$ of $K_1$ into $K$.

In addition, suppose that the mesh of $K$ is less than $\epsilon/2n$, where $\epsilon$ is a prescribed positive number We shall construct a sequence of simplicial mappings

$$\tilde{S}_0, \tilde{S}_1, \cdots, \tilde{S}_n$$

of the polyhedron $P = \| K \|$ into itself such that

a) the mapping $S_r$ $(0 \leq r \leq n)$ is a simplicial mapping of a subdivision $K^{(r)}$ of $K_1$ into $K$, all of whose fixed simplexes have dimension $\geq r$;

b) $$\rho(\tilde{S}_r, \tilde{S}_{r+1}) < \epsilon/n.$$

If we construct such a sequence, then $\tilde{S}_n = \tilde{S}$ satisfies conditions 1) and 2) of Theorem 3.3: all the fixed simplexes of $S_n$ have dimension $n$ and the inequality

$$\rho(\tilde{S}_n x, Cx) < n(\epsilon/n) = \epsilon$$

is satisfied for every $x \in P$.

The mapping $\tilde{S}_0$ and the subdivision $K^{(0)}$ of $K_1$ are given. We suppose that the mapping $\tilde{S}_r$ and the subdivision $K^{(r)}$ have been constructed and construct $\tilde{S}_{r+1}$ and $K^{(r+1)}$. However, we must preface the construction with several introductory remarks. We call a simplex $T$ of a subdivision $K'$ of $K$ a *principal simplex* (relative to $K$) if the dimension of its carrier in $K$ is equal to the dimension of $T$. If $K''$ is a subdivision of $K'$, then every simplex $T \in K''$, which is a principal simplex relative to $K$, is obviously a principal simplex relative to $K'$ and its carrier in $K'$ is a principal simplex relative to $K$. Finally, we note that every fixed simplex of a mapping of $K'$ into $K$ is a principal simplex relative to $K$.

LEMMA 3.51. *If $K^{(r)}$ is a barycentric subdivision of a triangulation $K$ of order $r$ ($r \geq 1$), the stars $O(K^{(r)}; T_1^s)$ and $O(K^{(r)}; T_2^s)$ of two arbitrary simplexes $T_1^s$ and $T_2^s$ of equal dimensions, which are principal simplexes relative to $K$, are disjoint.*

*Proof.* Lemma 3.51 can be stated as

3.510. Two simplexes of $K^{(r)}$ of the same dimension $s$, which are principal simplexes relative to $K$, cannot be faces of the same simplex of $K^{(r)}$.

If $K^{(r)}$ is a subdivision of $K'$ and $K'$ is a subdivision of $K$, then every simplex of $K^{(r)}$ which is principal relative to $K$ is a principal simplex relative to $K'$. Therefore, it is sufficient to prove 3.510 on the supposition that $K^{(r)}$ is the first order barycentric subdivision of $K$.

But then 3.510 follows from the fact that the only $s$-face of a simplex $(T_s^{q_p} > \cdots > T_p^{q_s} > \cdots > T_0^{q_0})$ of the barycentric subdivision of $K$ which can be a principal simplex is $(T_p^{q_s} > \cdots > T_0^{q_0})$.

We now proceed with the inductive construction which is the heart of the proof of Theorem 3.3.

Given $\tilde{S}_r$ and $K^{(r)}$, we are required to construct $\tilde{S}_{r+1}$ and $K^{(r+1)}$. Let $T_i^r$ ($1 \leq i \leq k$) be all the fixed $r$-simplexes of $S_r$. We construct a subdivision $K^{(r+1)}$ of $K^{(r)}$ by subdividing all the stars $O(K^{(r)}, T_i^r)$ in a special way; the mapping $S_{r+1}$ is obtained by changing $S_r$ only on these (pairwise disjoint!) stars.

The subdivision of $K^{(r)}$ is defined as follows: First, we construct the central subdivision of each fixed simplex $T_i^r$ relative to its boundary (which is not subdivided) (see IV, 4.3); next we construct the central subdivision of each $(r + 1)$-simplex $T^{r+1} \in O(K^{(r)}; T_i^r)$ relative to its boundary (one simplex of its boundary, namely $T_i^r$, has already been subdivided). After the $(r + s)$-simplexes have been subdivided, we construct the central subdivision of each $(r + s + 1)$-simplex of the star $O(K^{(r)}; T_i^r)$ relative to the already constructed subdivision of its boundary. The construction is finished when all the $n$-simplexes of the stars $O(K^{(r)}; T_i^r)$ $(1 \leq i \leq k)$ have been subdivided.

We now define the simplicial mapping $S_{r+1}$ of $K^{(r+1)}$ into $K$.

1°. If a vertex $e^{(r+1)} \in K^{(r+1)}$ is also a vertex of $K^{(r)}$, we set $S_{r+1}e^{(r+1)} = S_r e^{(r+1)}$.

2°. Suppose that a vertex $e^{(r+1)} \in K^{(r+1)}$ is not a vertex of $K^{(r)}$. Then $e^{(r+1)}$ is the centroid of one of the simplexes of a star $O(K^{(r)}; T_i^r)$ and we consider two cases:

$2_1$°. The vertex $e^{r+1}$ is the centroid of a simplex $T \in O(K^{(r)}; T_i^r)$ different from $T_i^r$. Then we choose an arbitrary vertex $e$ of the simplex $S_r T_i^r \in K$ and set $S_{r+1}e^{(r+1)} = e$.

$2_2$°. The vertex $e^{(r+1)}$ is the centroid of $T_i^r$. Then we let $S_{r+1}e^{(r+1)} \in K$ be an arbitrarily chosen vertex of a simplex of the star $O(K; S_r T_i^r)$ which is not a vertex of the simplex $S_r T_i^r$. (There is always such a vertex since $r < n$ and $K$ is a pure complex.)

We now prove that the mapping $S_{r+1}$ of the set of all vertices of $K^{(r+1)}$ into the set of all vertices of $K$ maps the star of every simplex $T \in K^{(r+1)}$ into the star of a simplex of $K$. It is clearly sufficient to show this for the simplexes $T$ of the subdivided stars $O(K^{(r)}; T_i^r)$. We consider two cases separately:

1) A vertex of $T$ is the centroid of some $T_i^r$; then the remaining vertices of $T$ are either vertices of $T_i^r$ or centroids of simplexes of $O(K^{(r)}; T_i^r)$; $S_{r+1}$ maps all these vertices into vertices of the simplex $S_r T_i^r$, and it maps the centroid of $T_i^r$ into a vertex $e \in O(K; S_r T_i^r)$. Hence, all the vertices of $T$ are mapped into vertices of the simplex

$$(eS_r T_i^r) \in O(K; S_r T_i^r), \quad (eS_r T_i^r) \in K.$$

2) No vertex of $T$ is the centroid of a simplex $T_i^r$; then the vertices of $T$ are either centroids of other simplexes of $O(K^{(r)}; T_i^r)$ or vertices of a simplex $T'$ of the boundary of this star; $S_{r+1}$ maps all these vertices into vertices of the simplex $S_r T_i^r$ and of the simplex $S_r T' \in |O(K; S_r T_i^r)|$ [$|O(K; S_r T_i^r)|$ is the combinatorial closure of $O(K; S_r T_i^r)$; see IV, Def. 1.32]. Hence all the vertices of $T$ are mapped into vertices of the simplex

$$(S_r T_i^r T') \in |O(K; S_r T_i^r)| \subset K.$$

Remark. A consequence of the above is

3.511. Let $O_i$ be the body of the star $O(K^{(r)}; T_i^r)$ and let $\Gamma_i$ be the body of the star $O(K; S_r T_i^r)$. If a simplex $T \in K^{(r+1)}$ is contained in $O_i$, then $S_{r+1}T \subseteq \bar{\Gamma}_i$.

We now prove that the simplicial mapping $S_{r+1}$ has properties a) and b) (see the beginning of 3.5).

a) The mapping $S_{r+1}$ has no fixed simplexes of dimension less than $r + 1$. For, let $T'$ be a simplex of $K^{(r+1)}$ of dimension $\leq r$. If

$$T' \subset \| K^{(r+1)} \| \setminus \cup O_i ,$$

then $T'$ is obviously not a fixed simplex of $S_{r+1}$, since $S_{r+1}$ coincides with $S_r$ on $\| K^{(r+1)} \| \setminus O_i$ [$S_r$ has fixed simplexes only of dimension $\geq r$, and the fixed $r$-simplexes of $S_r$ are the simplexes $T_i^r \in O(K^{(r)}; T_i^r)$.] Suppose $T' \subset O_i$. If $T'$ is a fixed simplex of $S_{r+1}$, then it must be a principal simplex relative to $K$ and therefore also relative to $K^{(r)}$.

Since all the simplexes of $O(K^{(r)}; T_i^r)$ different from $T_i^r$ have dimension $> r$, the carrier of $T'$ in $K^{(r)}$ must be $T_i^r$; hence, the dimension of $T'$ is $r$, so that one of the vertices of $T'$ is the centroid of $T_i^r$.

Since $T_i^r$ is a fixed simplex of $S_r$, $T_i^r \subset S_r T_i^r \in K$; hence

$$T' \subset T_i^r \subset S_r T_i^r \in K, \tag{3.52}$$

that is, the carrier of $T'$ in $K$ is $S_r T_i^r$. Therefore, if $T'$ were a fixed simplex of $S_{r+1}$, we would have

$$S_{r+1} T' = S_r T_i^r .$$

This is impossible, since $S_{r+1}$ maps one of the vertices of $T'$, namely the centroid of $T_i^r$, into a vertex of $K$ which is not a vertex of $S_r T_i^r$.

b) $\rho(\tilde{S}_r p, \tilde{S}_{r+1} p) < \epsilon/n$ for arbitrary $p \in \| K \|$. To show this, we note that $\tilde{S}_{r+1}$ differs from $\tilde{S}_r$ only on the sets $O_i$. But according to the definition of the sets $O_i$ and $\Gamma_i$ (see 3.511), $\tilde{S}_r p \in \Gamma_i$ for artibrary $p \in O_i$. Hence, by 3.511, $\tilde{S}_{r+1} p \in \bar{\Gamma}_i$. Since mesh $K < \epsilon/2n$ (see the beginning of 3.5),

$$\delta(\bar{\Gamma}_i) < \epsilon/n \quad \text{and} \quad (\tilde{S}_r p, \tilde{S}_{r+1} p) < \epsilon/n,$$

where $\delta(\bar{\Gamma}_i)$ is the diameter of $\bar{\Gamma}_i$. This completes the proof.

**§3.6. Remarks on the algebraic number of fixed points.** The algebraic number of fixed points of a mapping $C$ is, of course, not to be confused with the number of fixed points of $C$. If we denote the latter number by $N_C$, then $N_C = | \Lambda_C |$ if, and only if, all the fixed points have index $+1$ or if they all have index $-1$. If all the fixed points are *simple*, that is, if the index of each fixed point is equal to $\pm 1$, then we may assert only that $N_C \geq | \Lambda_C |$; since if $\pi$ is the number of fixed points of index $+1$ and $\nu$ is the number of fixed points of index $-1$, then $\Lambda_C = \pi - \nu$, $N_C = \pi + \nu$. We may call this case the "general" case, where the term is to be taken in its literal sense; on the other hand, we may call the case in which the index $\neq \pm 1$ "exceptional". The Remark at the end of 2.2 offers some justification for this distinction. Therefore, there is some basis in fact for asserting that *the number of fixed points of a continuous mapping is, "in general", not less than the absolute value of the algebraic number of fixed points.*

We must point out, however, that the exact number of fixed points of a particular continuous mapping is an accidental property of the mapping: in a topological theory of continuous mappings our central interest does not lie in the properties peculiar to each individual mapping, but *in the properties common to a whole class of homotopic (or even homologous) mappings.* The degree of a mapping and the algebraic number of its fixed points are examples of such properties. It is therefore natural to ask for *the minimum number of fixed points which are possible in a prescribed homotopy (or homology) class of mappings.* We denote this number by $\lambda_C$ (or $\lambda_C'$), where $C$ is a mapping of the given homotopy (homology) class. Obviously, $N_C \geq \lambda_C \geq \lambda_C'$ for arbitrary $C$.

Unfortunately, little is known about the numbers $\lambda_C$ and $\lambda_C'$, except for the fact that they are finite for each class derived from Theorem 3.3. It is true that $\Lambda_C \neq 0$ implies $\lambda_C \geq \lambda_C' > 0$. On the other hand (see Theorem 1.5), it is possible to have $\lambda_C' > \Lambda_C = 0$. For a mapping of a circumference into itself of degree $c$, $\Lambda_C = 1 - c$, $\lambda_C = |1 - c|$; a mapping of a closed simplex into itself yields $\lambda_C = \Lambda_C = 1$.

If we compactify the plane with a point at infinity, to obtain a sphere $S^2$, and consider a translation of the plane (the point at infinity is the only fixed point under this transformation), we obtain a mapping $C$ of $S^2$ onto itself of degree 1. Then $\lambda_C = 1$, $\Lambda_C = 2$ and, consequently, $\lambda_C < \Lambda_C$. The problem of investigating the number $\lambda_C$ is interesting and difficult. Nielsen has devoted many papers to this subject; his methods are essentially different from those presented in this book. But, at any rate, the algebraic number of fixed points gives the first, and up to the present the most important, estimate of the number of fixed points of a continuous mapping.

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# INDEX

Alexander duality, combinatorial form, 24
Alexander duality, combinatorial form (mod $m$), 25
Alexander duality for polyhedra, 53
Alexander duality for polyhedra (mod $m$), 53
Alexander-Pontryagin duality, 52
Alexander-Pontryagin duality for finite compacta, 54ff.
algebraic number of fixed points, 135, 138
algebraic number of fixed $r$-simplexes, 128, 129
approximation theorem, 140

Barycentric subdivision of a chain, 50ff.
body of a chain, 50
boundary of an oriented polyhedral domain, 106
Brouwer, 100

Carrier of a simplex, 60
characteristic of a normal vector field on a sphere, 112, 113
characteristic of a vector field, 112
classification of continuous mappings, 100
cofinal suborder of a spectrum, 46
cohomology classes of cocyles, 42
cohomology groups of bicompacta, 45
cohomology groups of compacta, 41, 44
cohomology groups of finite compacta, 49
cohomology of cocycles in a spectrum, 41ff.
combinatorial $h$-manifold, 4
constant mapping, 120

Degree of a mapping in a point, 107
degree of a topological mapping, 109, 110
$\nabla^{00}$-classes, 45, 46
$\nabla$-classes of cocycles, 42
$\nabla$-groups of bicompacta, 45
$\nabla$-groups of compacta, 41
directed partially ordered set, 41
direction field, 114
displacement field, 134
displacement vector, 134

Euler characteristic of an $h$-manifold, 20
existence of fixed simplexes, 130
existence of roots, 105, 106
extensible cocycle, 26ff.
extension theorem, 120, 124
exterior of a closed pseudomanifold, 94

Fixed points and homotopy, 117
fixed points of affine mappings, 138
fixed simplex, 128
Frobenius' theorem, 119
full projective spectrum, 45

Generalized intersection number, 78, 79, 80
group $\nabla^{00}$, 45, 46
group $^*Z^r$, 28

$h$-manifold, 4
homological manifold, 4
homology groups of open subsets of $S^n$, 50
homology of proper cycles, 51
homotopic to zero, 120
Hopf, H., 100, 128, 129
Hopf's formula, 130
Hopf's theorem, 120

Index of a fixed point, 134
index of a point relative to a cycle, 100
index of a point relative to a proper cycle, 101, 102
index of a pseudomanifold relative to a point, 95
index of an isolated zero, 113
induced orientation, 104
interior of a closed pseudomanifold, 94
intersection number, 13, 21, 23, 73, 74, 79, 80
invariance of homology groups, 51
invariance of linking number under deformation, 87, 88
invariance of the index, 110
invariance of the local degree, 111
isomorphism $D_p$, 19
isomorphism $D^q$, 17
isomorphism $\Delta D^q$, 30

Jordan-Brouwer theorem, 54, 94
Jordan theorem, combinatorial form, 25

Kolmogorov, 2, 19

Lefschetz, 128
Lefschetz-Hopf fixed point theorem, 131
Lefschetz number, 131, 133
linked cycles, 81
linking and homology, 82
linking of proper cycles, 86
linking of pseudomanifolds, 95
linking number, 35, 81
linking number of pseudomanifolds, 95
little Alexander duality, 88
little Pontryagin duality, 91
local degree of a mapping, 106ff.

Mapping induced by a vector field, 112
mapping of a chain, 106
mappings of degree zero, 122
mesh of a complex, 57
minimum number of fixed points, 145

$n$-dimensional $h$-manifold, 4
natural homomorphism $\bar{S}^{\alpha}_{\cdot}$, 44
negative fixed simplex, 128
nerve of a star covering, 67
Nielsen, 145
nonorientable $h$-manifold, 6
normal displacement of a chain, 51
normal fixed point, 136
normalizable fixed point, 136
number of fixed points, 144

Orientable $h$-manifold, 6
oriented fixed simplex, 128

Poincaré-Bohl theorem, 103
Poincaré-Brouwer theorem, 116
Poincaré duality, 19, 20
Poincaré duality (mod $m$), 20
Poincaré space, 7
polyhedral domain, 106
Pontryagin, 2, 6, 37, 38
positional topological invariant, 96
positive fixed simplex, 128
principal simplex, 60
projections, 41
projective spectrum, 41
projective spectrum of a bicompactum, 45, 46
projective spectrum of a polyhedron, 48, 49
proper cycle of an open subset of $S^n$, 51

Reflections of spheres, 117
regular fixed point, 138, 139
relative general position of chains, 73
root of a mapping, 106
Rouché's theorem, 103

Simple $h$-manifold, 6
simple fixed point, 144
singularity of a direction field, 114
spectrum, 41
spectrum of a compactum, 68
sphere of directions, 114
star covering, 66, 67
suborder of a partially ordered set, 46
systems of linked cycles, 37
systems of linked proper cycles, 87

Topological invariance of the index of a fixed point, 137

Veblen's theorem, 23
vector field, 113
vector fields on a solid sphere, 114, 118, 119

Whitney, 120

Zeros of vector fields, 113